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Interplay between the Geometry of a Riemannian Manifold and the Geometry of Its Isometries

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Abstract-In this paper, the expression of a Riemannian manifold M by a suitable fibration in the letters of the quotients of its isometries $I(M)$ and its slices as the fiber has been shown. Also, the explicit corresponding relation between their Lie algebras has been driven. On the other hand, by another glance to tangent bundle $T(M)$ as an associated bundle to the frame bundle M, $L(M)$, and via a Lie algebra homomorphism between the Lie algebras of $T(M)$, $\mathcal{L}(I(M))$ and $\chi(M)$, the main tools of the geometry of $T(M)$ to the ones of M has been translated. Lastly, an exact sequence of vector bundles and Lie algebra sheaves corresponded to homogeneous bundle $I(M)$ with the fibration of its slices, which is an example of "Atiyah sequence" has been extended.

Keywords- *Isometries, Associated Bundles, Proper Actions, Lie Algebra Homomorphisms, Slices*

I. INTRODUCTION

The impression of algebraic properties of isometries of a Riemannian manifold M, $I(M)$ on the geometric properties of it, can be found in old text such as [1-2]. In recent works, implicit use of this impression can be found in [3-4] for normal homogeneous manifolds, also, by passing to Killing vector fields of constant length in [5-7] for pseudo Riemannian manifolds of constant curvature. Therefore, the clarification of this connection for Riemannian manifolds is an approval of related concepts through various works and somewhat simplifying of them, too, which we progress it in a general setting.

This progress consists of three parts: In Section 2, in a pure setting, contains the clarification of geometric and algebraic properties of isometries. This results to the expression of M in letters of the action $I(M)$, and its slices at a fixed point of this action and then, this identification in the literature of Lie algebras, mentioned in Theorems 3, 10, 15 and Corollaries 5, 6, 12, 13, 16, and Prologue to a Rigidity Problem, Problem 14.

In Section 3, with an applied approach, we use of a suitable Lie algebra homomorphism between $I(\mathcal{M})$ and $\chi(M)$, then let $T(M)$ as an associated bundle to the frame bundle $L(M)(M, GL(M,\mathbf{R}))$ and consider the corresponded equivariant functions. In this way, the main tools of the geometry of M, in related to $\chi(M)$, can be translated to the letters of $I(\mathcal{M})$, mentioned in Theorem 18 and Corollaries 18, 20. Lastly, in Section 4, this progress reach us to an exact sequence of vector bundles and of Lie algebra sheaves as an example of Atiyah sequence which is the most sequence in Lie groupoid theory. It is to be noted that some rigidity problems in the geometry of Lie groups can be solved by Lie groupoid notions, for example refer to [8-10].

II. SOME FUNDAMENTAL RESULTS

Let A be a smooth action of a Lie group G on M which is proper at a point $x \in M$ which means that there exists a neighborhood U of x in M such that $\{g \in G \mid A(g)(U) \cap U \neq \phi\}$ has a compact closure in G, so G_x is a compact subgroup of G. A subset N of M is said to be G-invariant if $g.y \in N$, whenever $y \in N$, $g \in G$, equivalently N is a union of orbits. If A, and A' be actions of G on the manifolds M and M', respectively, then a smooth mapping $\Phi: M \to M'$ intertwines A with A', or is G-equivariant of M to M', if $\Phi \circ A(g) = A'(g) \circ \Phi$, $\forall g \in G$. Also, Φ is an equivalence of smooth actions if it is a smooth diffeomorphism of M to M' intertwining A with A'. In this case, the actions A and A' are said to be smooth equivalent.

We consider the smooth action $A_m: G \to M$ at a point $m \in M$, fixed in the Part I, and the infinitesimal action $\alpha_m: T_e A_m: \mathcal{G} \to T_m M$ at this point:

Proposition 1. ([11]) Let A be a smooth action of a Lie group G on a manifold M and proper at a point $m \in M$. Then there exists a G-invariant open neighborhood U of m in M such that the G-action in U is smooth equivalent to the action of G on $G \times_{G_m} E$ with an equivalence $\Phi: G \times_{G_m} E \to U$.

Here E is an open G_m-invariant neighborhood of 0 in $T_m M / \alpha_m(\mathcal{G})$, on which G_m acts linearly, via the tangent action $k \to T_m A(K)$ modulo $\alpha_m(\mathcal{G})$.

Remark 2. In explicit words, Proposition 1 says that in a suitable G -invariant neighborhood of any orbit $G.m$, the action is equivalent to a standard one that is constructed in the terms of the Lie group G , the stabilizer group G_m , and the tangent representation of G_m on $T_m M / T_m(G.m)$, that this structure explained in framework of associated fiber bundles.

For constructing an associated fiber bundle, we consider $P(M,G)$ as a principal fiber bundle and F as a left G -manifold, i.e., G acts by diffeomorphisms from the left on F . We denote by r this action of G on F and by gf or $r(g)f$ the element $r(g,f)$. Then we have the following right action R of G on the product manifold $P \times F$, $(u,f)g = (ug, r(g^{-1})f$, $\forall g \in G, (u,f) \in P \times F$. The action R is free and its orbit space is denoted by $P \times_r F$ or by $E = E(M,F,r,P)$. Precisely, we have the following commutative diagram:

$$\begin{array}{ccc} P \times F & \to & P \\ \downarrow & & \downarrow \\ E & \to & M \end{array}$$

Diagram 1

where $[,]: P \times F \to E$ defined by the rule $(u,f) \to [u,f]$ and $\pi_E : E \to M$, by the projection $\pi : E \to M$, i.e., $\pi_E([u,f]) = \pi(u)$, also the map $P \times F \to P$, is the projection onto the first factor. If the action of G on P and F is considered to be proper too, then the action of G on $P \times F$, will be proper.

Theorem 3. ([11]) Let G be a Lie group acting properly and in a smooth fashion on the (paracompact) manifold M . Then M has a G -invariant smooth Riemannian structure g .

Remark 4. Theorem 3 is the other interpretation of this famous result: "Every reduction of the structure group $GL(m,\mathbf{R})$ of $T(M)$ to $O(m)$ gives rise to a Riemannian metric g on M , and conversely."

Corollary 5. As the before notions, there is a connection between $L(M)$ and $I(M)$.

Proof. Let $O(M)$ be a compact subgroup of $GL(m,\mathbf{R})$ acting by linear transformations on a finite dimensional vector space $\mathbf{R}^m$. By taking an arbitrary left invariant Riemannian structure α on $GL(m,\mathbf{R})$ and some inner product β on $\mathbf{R}^m$, we get a left $GL(m,\mathbf{R})$ -invariant Riemannian structure $\gamma = \alpha \times \beta$ on $GL(m,\mathbf{R}) \times \mathbf{R}^m$. Then, by averaging over the action of $O(M)$ on $GL(m,\mathbf{R}) \times \mathbf{R}^m$, we get a real-analytic, $O(M)$ -invariant Riemannian structure $\bar{\gamma}$ on $GL(m,\mathbf{R}) \times \mathbf{R}^m$ which still is left $GL(m,\mathbf{R})$ -invariant.

In this way, by the projection map $GL(m,\mathbf{R}) \times \mathbf{R}^m \to GL(m,\mathbf{R}) \times_{O(M)} \mathbf{R}^m$, we reach to a unique Riemannian structure γ on $GL(m,\mathbf{R}) \times_{O(M)} \mathbf{R}^m$ which is real-analytic and $GL(m,\mathbf{R})$ -invariant. So, as Proposition 1, this Riemannian structure is mapped by the smooth equivalence to a smooth $GL(m,\mathbf{R})$ -invariant Riemannian structure on the $GL(m,\mathbf{R})$ -invariant open neighborhood U of $m \in M$. Finally, it is sufficient to use a $GL(m,\mathbf{R})$ -invariant partition of unity subordinate to the U 's of the smooth class. In effect, we should consider this fact that the Riemannian metric on a manifold M is a Riemannian metric on $TM(M, \mathbf{R}^m, r, L(M))$ as an associated bundle to the frame bundle $L(M)(M, GL(m,\mathbf{R}))$.

In the sequel, this connection will be expressed in the clearer letters.

Corollary 6. The isotropy representation $\rho : T_m M \to GL(T_m M)$; $\rho(g) = T_m g$, defines an isomorphism of $(I(M))_m$ onto a closed subgroup of $O(T_m M, g_m) \subset GL(T_m M)$.

Proof. We use this fact that the action of $GL(m,\mathbf{R})$, restricted to a suitable $GL(m,\mathbf{R})$ -invariant open neighborhood U in M of the fixed point m , is equivalent to the linear tangent action of G on $T_m M$, restricted to an open neighborhood of 0 in $T_m M$.

Also, injectivity of linear isotropy representation is by this fact that an isometry g is determined by giving only the image $g(x)$ of a point x and the corresponding tangent map $T_x g$.

Remark 7. More interesting result than Theorem 3 is, conversely:

If g is a smooth Riemannian structure on a manifold M , then the group $I(M)$ of isometries of the corresponding metric space is equal to the group of automorphisms of (M,g) , and is a finite-dimensional Lie group. Its action on M is proper and smooth, and its Lie group topology coincides with the smooth topology on $I(M) \subset Diff^{\infty}(M)$, and with the topology of pointwise convergence. Then, any closed subgroup G of $I(M)$ is also a Lie group acting properly and in a smooth fashion on M . Also, the orbits of G are closed if and only if the action of G is orbit equivalent to the action of the closure of G in $I(M)$.

Example 8. The canonical sphere S^n may be viewed as the homogeneous manifold $SO(n+1)/SO(n)$, but there also exist other compact connected Lie groups acting effectively and transitively on some spheres. Table 1 and 2 can be found in the works of Borel ([12]) and Montgomery ([1]). The action of G , as a closed subgroup of $I(M)$ on the corresponding sphere S^k is obtained by considering some special linear representation of G in $\mathbf{R}^{k+1}$ such that G acts transitively on the unit sphere of $\mathbf{R}^{k+1}$ (Corollary 6 and Remark 7).

TABLE I. SPHERES AS A HOMOGENEOUS MANIFOLD

G	Sphere	G_m
$SO(n)$	S^{n-1}	$SO(n-1)$
$U(n)$	S^{2n-1}	$U(n-1)$
$SU(n)$	S^{2n-1}	$SU(n-1)$
$Sp(n)Sp(1)$	S^{4n-1}	$Sp(n-1)Sp(1)$
$Sp(n)U(1)$	S^{4n-1}	$Sp(n-1)U(1)$
$Sp(n)$	S^{4n-1}	$Sp(n-1)U(1)$

and some special cases:

TABLE II. SPECIAL CASES OF TABLE 1

G	Sphere	G_m
G_2	S^6	$SU(3)$
$Spin(7)$	S^7	G_2
$Spin(9)$	S^{15}	$Spin(7)$

Definition 9. A smooth slice at $m \in M$ for the smooth action A on M is a smooth submanifold S of M through m such that, in before notations:

i. $T_m M = \alpha_m(\mathcal{G}) \oplus T_m M$ & $T_x M = \alpha_x(\mathcal{G}) + T_x S$ $(\forall x \in S)$,

ii. S is G_m-invariant,

iii. if $x \in S$, $g \in G$ and $A(g)(x) \in S$, then $g \in G_m$

Theorem 10. For the smooth action $I(M)$ on M and proper at $m \in M$, we get the $I(M)$-equivariant smooth fibration (fiber bundle), $\rho : U \to I(M)/(I(M))_m$, where U is the $I(M)$-invariant open neighborhood of $m \in M$ as mentioned in Proposition 1, and we will have the trivialization

$U \cong I(M)/(I(M))_m \times F,$

with the fiber F which is the slice at $m \in M$ for the smooth action of $I(M)$ on M (at least, up to isomorphism).

Proof. As Proposition 1, for the smooth action $G = I(M)$, we observe that

$\Phi^{-1} : U \to I(M) \times_{(I(M))_m} E$

followed by the projection

$I(M) \times_{(I(M))_m} E \to I(M)/(I(M))_m$,

defines a smooth $I(M)$-equivariant fibration $U \to I(M)(M)/(I(M))_m$, for which the orbit $I(M).m$ is a global section. For the last assertion, we come back to details of the fibration: In reality, E is the smooth slice at $m \in M$ for the smooth action A of $I(M)$ on M, since:

i. $T_m M = \alpha_m(\mathcal{I}(M)) \oplus T_m E, T_x M = \alpha_x(\mathcal{I}(\mathcal{M})) \oplus T_x E$ $(\forall x \in E)$

ii. E is $(I(M))_m$-invariant.

iii. If $x \in E$, $g \in I(M)$ and $A(g)(x) = g(x) \in E$, then $g \in (I(M))_m$.

Also, the equivalence map

$\Phi : I(M) \times_{(I(M))_m} E \to U$

is defined as $A|_{I(M) \times E} = \Phi \circ \pi$ such that

$\pi : I(M) \times E \to I(M) \times_{(I(M))_m} E$

is the principal fiber bundle with the structure group $(I(M))_m$ which leaves invariant E, and of course, Φ conserves the fibers. The projection

$I(M) \times_{(I(M))_m} E \to I(M)/(I(M))_m$

is the smooth fibration with the fiber E, too. In summation of the discussions, $F = E$ (at least, up to isomorphism).

Remark 11. The properness of the $I(M)$-action on

$I(M) \times_{(I(M))_m} E$

implies that the action of $I(M)$ on the $I(M)$-invariant open neighborhood U of m in M is proper, too. Roughly speaking, proper at m is equivalent to proper on one $I(M)$-invariant neighborhood.

Corollary 12. The fiber F in Theorem 10, is the submanifold of M which intersects $I(M).m$ transversely and has complementary dimension.

Proof. The map $I(M)/(I(M))_m \to I(M).m$ is a bijective map and in reality, a smooth immersion which exhibiting the orbits as an immersed smooth submanifold of M. On the other hand, the identity map of E into M induces a homeomorphism from $(I(M))_m$-orbits in E, $E/(I(M))_m$, onto an open neighborhood of $I(M).m$ in the space of $I(M)$-orbits in M. Then, the assertion is a direct result of Definition 9.

These dependence between $I(M)$ and M does not conclude to this results. In the setting of Lie algebras, by considering Theorem 2, we will have, too:

Corollary 13. For the transitive and proper actions, the identification of $\mathcal{I}(\mathcal{M})$ gives rise to the identification of $\mathcal{M}$ and conversely. Then, Killing vector fields can be recognized of $\mathcal{M}$ and conversely.

Proof. The Killing vector fields at a point $m \in M$ are tangent to the orbit of this point and corresponds to $\mathcal{I}(\mathcal{M})$ under an anti-isomorphism Lie algebra. Then, it is sufficient to use of the relation, $T_m M = \alpha_m(\mathcal{I}(\mathcal{M})) \oplus T_m E$, which $T_m E$ is zero, for these actions.

In more precise letters, as mentioned in Introduction, we will have:

Problem 14 (Prologue to a Rigidity Problem). Let M, M' are two m -dimensional homogeneous Riemannian manifolds, and f is a continuous and group isomorphism map of $I(M)$ to $I(M')$. Let $V \in \chi(M)$ (the (complete) Killing vector fields of M). Then, we get to a vector $v' \in T_{m'}M'$ corresponded to V , under a homomorphism of $\chi(M)$ onto $T_{m'}M'$ (for $m' \in M'$). Also, we reach to a Lie anti-isomorphism between $\chi(M)$ and $\mathcal{I}(\mathcal{M}')$.

Answer. Let

$$\{\Phi_t\}_t \in I(M)$$

is the (full) flow of the (complete) vector field $V \in \chi(M)$. We consider the unique flow

$$\{\Psi_t\}_t \in I(M')$$

corresponded to it by the map f . Then, we get to the Killing vector field $\tilde{V}' \in \chi(M')$, as an element of the Lie algebra corresponded to its one-parameter subgroup $t \to \Psi_t$. Lastly, we get to $V' \in \mathcal{I}(\mathcal{M}')$ under the Lie anti-isomorphism $\mathcal{I}(\mathcal{M}') \to \chi(M')$, and project it to a $v' \in T_{m'}M'$ under the projection $d\pi$; note that $\pi : I(M') \to M'$ is submersion because of transitivity of the action $I(M')$ on the (homogeneous) manifold M' .

Also, for the flows of $V, W \in \mathcal{I}(\mathcal{M})$ shown by φ, ψ at a point $x \in M$ (respectively), it can be proved that Λ as the flow of $[V, W]$ equals the following:

$$\Lambda(t,x) = \lim_{n\to\infty}\Big\{\Big(\psi_{-\sqrt{t/n}} \circ \varphi_{-\sqrt{t/n}} \circ \psi_{-\sqrt{t/n}} \circ \varphi_{-\sqrt{t/n}}\Big) \circ \cdots \circ \Big(\psi_{-\sqrt{t/n}} \circ \varphi_{-\sqrt{t/n}} \circ \psi_{-\sqrt{t/n}} \circ \varphi_{-\sqrt{t/n}}\Big)\Big\}(x)$$

thus, f respects the corresponding brackets. The structure of the proof has been shown in Dig.2:

$$\begin{array}{ccccc}
V \in \chi(M) & & \tilde{V}' \in \chi(M') & \leftrightarrow & V' \in \mathcal{I}(\mathcal{M}') \\
\downarrow & & \uparrow & & \downarrow \\
\{\Phi_t\}_t \in I(M') & \leftrightarrow & \{\Psi_t\}_t \in I(M') & & v' \in T_{m'}M'
\end{array}$$

Diagram 2

In completion of the relation between $I(M)$ and M in Lie algebra letters, we have:

Theorem 15. Let K be a smooth submanifold of $I(M)$, through its identity e such that $\mathcal{I}(\mathcal{M}) = (\mathcal{I}(\mathcal{M}))_m \oplus T_eK$ where

$$(\mathcal{I}(\mathcal{M}))_m = \ker\alpha_m = \{X \in \mathcal{I}(\mathcal{M}) \mid \alpha_m(X) = 0\}.$$

Then, there exist open neighborhoods $\hat{K}$ and $\hat{E}$ of e and m in K and the slice E , respectively, such that

$$A|_{\hat{K}\times\hat{E}}$$

smooth diffeomorphism from $\hat{K}\times\hat{E}$ onto an open neighborhood $\hat{M}$ of $m \in M$.

Proof. We have

$$T_{(e,m)}A|_{K\times E} : T_eK \times T_mE \to T_mM;$$
$$T_{(e,m)}A|_{K\times E}\,(X, v) = \alpha_m(X) + v$$

Because of $\alpha_m(\mathcal{I}(\mathcal{M})) \cap T_mE = 0$, $\alpha_m(X) + v = 0$, the equation $\alpha_m(X)$ results in $\alpha_m(X) = 0$ and $v = 0$. Next, $(\ker\alpha_m) \cap T_eK = 0$ implies that $X = 0$. Then this map is injective.

Also,

$$\begin{aligned}\dim(K\times E) &= \dim I(M) - \dim\ker\alpha_m + \dim T_mM - \dim\alpha_m(\mathcal{I}(\mathcal{M})) \\ &= \dim T_mM = \dim M\end{aligned}$$

So, it is sufficient to apply the inverse mapping theorem.

Corollary 16. For infinitesimally locally free action at m , $I(M)$ acts locally on M by the multiplication only on the first factor, i.e., $\hat{K}$, as Theorem 15.

Proof. In this case, $(\mathcal{I}(\mathcal{M}))_m = 0$. Then, $\hat{K}$ is an open neighborhood of e in $I(M)$. Then, in the local identification of M with $\hat{K}\times\hat{E}$, the local action of $g \in I(M)$ consists of (left) multiplication by $\mathcal{I}(\mathcal{M})$ only on the first factor.

III. SOME APPLIED RESULTS

After the discussion, we want to exit of this abstract setting and, in a applied manner, the main concepts of the geometry of $I(M)$ express in the ones of M . Thus, for this translation we seek a suitable Lie algebra homomorphism between $\mathcal{I}(\mathcal{M})$ and $\mathcal{M}$, (as analytic vector fields on identity) and maybe its extension, the Lie algebra of vector fields on M :

By considering the proper action of $I(M)$ on M , $\mathcal{I}(\mathcal{M})$ can be connected to the Lie algebra of vector fields of M , $\chi(M)$. At first, in general case, if the Lie group G acts on M (on the right), we assign to each element $V \in \mathcal{G}$ a vector field $V^* \in \chi(M)$ by restricting the action to the 1-parameter subgroup $\gamma(t) = \exp(tV)$ on M . Then, we have the following lemma:

Lemma. ([3]) Let a Lie group G act on M (on the right). The mapping $F : \mathcal{G} \to \chi(M)$ which sends V into V^* is a Lie algebra homomorphism. If G acts effectively on M , the F is an isomorphism of $\mathcal{G}$ into $\chi(M)$. If G acts freely on M , then, for each non-zero $V \in \mathcal{G}$, $F(V)$ never vanishes on M .

Note that $\chi(M)$ is the section of $T(M)$, and $TM\left(M,\mathbf{R}^m,r,L(M)\right)$

is the associated bundle to $L(M)(M,GL(m,\mathbf{R}))$ by fiber type $\mathbf{R}^m$ and the left action r given by the representation of $GL(m,\mathbf{R})$ on $\mathbf{R}^m$. Precisely by substituting in the Dig.1 of Section 2, we get into the following diagram:

$$\begin{array}{ccc} L(M)\times\mathbf{R}^m & \rightarrow & L(M) \\ \downarrow & & \downarrow \\ T(M) & \rightarrow & M \end{array}$$

Diagram 3

For $u\in L_x(M)$ we have the map $\tilde{u}\in\mathbf{R}^m\rightarrow T_xM$, which is a linear isomorphism. In the other words, we can define a frame by the map $\tilde{u}$ using the vector bundle structure of TM with the action of $GL(m,\mathbf{R})$ given by the rule $\tilde{u}.g=\tilde{u}\circ g$, where $g\in GL(m,\mathbf{R})$ is regarded as a map from $\mathbf{R}^m$ to $\mathbf{R}^m$.

Then, we reach to a one-to-one correspondence between $GL(m,\mathbf{R})$-equivariant functions from $L(M)$ to $\mathbf{R}^m$,

$F_{GL(m,\mathbf{R})}(L(M),\mathbf{R}^m)$,

and the sections of TM, $\chi(M)$, which is defined as follows:

Let $V\in\chi(M)$, then we define

$$f_V: L(M)\rightarrow\mathbf{R}^m;\ f_V(u)=\tilde{u}^{-1}(V(x))$$

where $\pi(u)=x$ and $\tilde{u}:F\rightarrow E_x$ is the isomorphism defined by $u\in L(M)$. Conversely, given a $GL(m,\mathbf{R})$-equivariant

$f\in F_{GL(m,\mathbf{R})}(L(M),\mathbf{R}^m)$,

i.e., $f(ug)=g^{-1}f(u)$, ($\forall g\in GL(m,\mathbf{R})$, $u\in L(M)$) then, we define

$V_f\in\chi(M)$

by

$$V_f(x)=[u,f(u)],$$

as the notions of Section 2, where $u\in\pi^{-1}(x)$. This is well-defined because of equivariance.

Theorem 18. The covariant derivative restricted to the elements of $\mathcal{I}(\mathcal{M})$, of any order, can be defined in the terms of Lie derivative of vector fields on M, and also, in the letters of canonical forms of $L(M)$.

Proof. The covariant derivative D_UV corresponds to

$L_{\hat{U}}(f_V)$

where f_V defined as above notions, and $\hat{U}$ is the horizontal lift of $U\in\chi(M)$ to $L(M)$. Precisely, $D_UV=\tilde{u}(\hat{U}.f_V)$. Also,

$$(D(U))x=\tilde{u}(\hat{U}_x(\theta(\hat{V})));\ \pi(u)=x$$

and $U,V\in\chi(M)$ which θ is the canonical form of $L(M)$, i.e., the $\mathbf{R}^m$-valued 1-form on $L(M)$ defined by

$$\theta(X)=\tilde{u}^{-1}(\pi_*(X)),\ \forall X\in T_uL(M)$$

Then, by considering $G=I(M)$ and its action on M in the sense of $x.g=g(x)$, ($\forall g\in I(M)$), we get a Lie algebra homomorphism between $\mathcal{I}(\mathcal{M})$ and $\chi(M)$ as Lemma 17. So, we get into the following diagram:

$$\begin{array}{ccc} \tilde{u}^{-1}((FV)_x)\in F_{GL(m,\mathbf{R})}(L(M),\mathbf{R}^m) & \leftrightarrow & (FV)_x\in\chi(M) \\ \downarrow & & \downarrow \\ V_e\in\mathcal{I}(\mathcal{M}) & & V_e\in\mathcal{I}(\mathcal{M}) \end{array}$$

Diagram 4

If the action A of $I(M)$ on M is effective (by passing to the Lie group $I(M)/\ker(A)$ if necessary), we can finally get a one-to-one linear mapping of $\mathcal{I}(\mathcal{M})$ to

$F_{GL(m,\mathbf{R})}(L(M),\mathbf{R}^m)$.

Finally, the covariant derivatives D, $D(D)$, … of any order can be translated to the corresponding letters.

Remark 19. This translation is very valuable, since the definition of the covariant derivatives in contrast with the Lie derivatives need to define the further construction, i.e., the connection on a manifold. Furthermore, the calculations by

$$L_{\hat{U}}(f_V)=df_V(\hat{U})=\hat{U}.f_V$$

are simpler than D_UV. Also, we need to pass of G-invariant functions because of the simplification of complex notions and of course, for well definiteness of these action-invariant notions.

Corollary 20. For $U,V\in\chi(M)$, $u\in L(M)$ (or for the elements of $\mathcal{I}(\mathcal{M})$ corresponded to them as Dig. 4),

$$f_{[u,v]}(u)=[f_U(u),f_V(u)].$$

Proof. As Lemma 17, $F:\mathcal{G}\rightarrow\chi(M)$ can be defined in this manner:

For every $x\in M$, let F_x be the mapping $a\in G\rightarrow xa\in M$, then $(F_x)_*V_e=(FV)_x$. It follows that F is a linear mapping of $\mathcal{G}(G)$ into $\chi(M)$. To prove that F commutes with the bracket, let $V,W \in \mathcal{G}$, $V^* = FV$, $W^* = FW$, $\gamma_t = \exp tV$. Then (by ignoring the details of computations),

$$[V^*, W^*] = \lim_{t\to 0}\frac{1}{t}\left(F_x W_e - F_x\left(ad\left(\gamma_t^{-1}\right)W_e\right)\right)$$
$$= F_x\left(\lim_{t\to 0}\frac{1}{t}\left(W_e - ad\left(\gamma_t^{-1}\right)W_e\right)\right)$$
$$= F_x\left([V,W]_e\right) = \left(F[V,W]\right)_x$$

On the other hand, for $U,V \in \chi(M)$ and $x \in M$, the value at x of the (Lie algebra) bracket $[U,V]$ is $[U(x),V(x)]_x$ where $[,]_x$ is the restriction of $[,]$ to associated bundle $\left(L(M)\times\mathcal{GL}(m,\mathbf{R})\right)/GL(m,\mathbf{R})|_x$, i.e., T_xM. Then we should show that for $u \in \pi^{-1}(x)$ and $X,Y \in \mathcal{GL}(m,\mathbf{R})$,

$$[[u,X],[u,Y]]_x = [u,[X,Y]],$$

which means that F is a Lie algebra homomorphism of $\mathcal{G}$ into $\chi(M)$.

In this way, since Ad_g is a Lie algebra automorphism for all $g \in GL(m,\mathbf{R})$, this bracket is well defined. In the sequel, the proof will be completed behind to the flows as above with noting that the bracket on the right hand side is the right hand bracket on $\mathcal{GL}(m,\mathbf{R})$.

Corollary 21. The curvature tensor field and torsion tensor field restricted to the elements of $\mathcal{I}(\mathcal{M})$, can be defined in the letters of $\chi(M)$, too.

Proof. At first, we prove that the corresponding k-forms, $k \geq 1$, can be translated: By the one-to-one correspondence mentioned in this section, for $\phi \in \wedge^k\left(L(M),\mathbf{R}^m\right)$ as a tensorial k-form of type $(ad,\mathcal{GL}(m,\mathbf{R}))$, (i.e., $R_a^*\phi = ad(a^{-1}).\phi$ $(\forall a \in GL(m,\mathbf{R}))$, and $\phi(X_1,...,X_k) = 0$, whenever some X_i; $i = 1,...,k$ is vertical), there exists a unique k-form S_ϕ on M with values in the vector bundle $T(M) = L(M)\times_{GL(m,\mathbf{R})}\mathbf{R}^m$ defined as follows,

$$S_\phi(x)\left(X_1,...,X_k\right) = \tilde{u}\phi(u)\left(Y_1,...,Y_k\right),\ \forall x \in M,$$

where $u \in \pi^{-1}(x)$ and $Y_i \in T_uL(M)$ such that $T\pi(Y_i) = X_i$, $i = 1,...,k$ (not necessarily horizontal).

Due to tensoriality of ϕ, this definition of S_ϕ is independent of the choice of u and Y_i. Now, it is sufficient to apply Dig.4 and use of definition of curvature and torsion tensor fields, (for more details about this notions, refer to [13]).

IV. AN EXTENSION

Of before two sections, we reach to one exact sequence for vector bundles and of Lie algebras sheaves, which high lights the connection between $I(M)$; $\mathcal{I}(\mathcal{M})$ and ones of M, as we wanted. In this way, we reach to an example of Atiyah sequence with the same algebraic and geometric properties, that it is the most sequence in Lie groupoid theory ([14]):

For the closed subgroup $(I(M))_m$ of the Lie group $I(M)$, we consider the homogeneous bundle

$$I(M)\left(\frac{I(M)}{(I(M))_m},(I(M))_m\right)$$

(locally on $I(M)$-invariant open neighborhood of m). Now we will reach to the sequence,

$$\frac{I(M)\times(\mathcal{I}(\mathcal{M}))_m}{(I(M))_m} \to \frac{T(I(M))}{(I(M))_m} \to T\left(\frac{I(M)}{(I(M))_m}\right) \tag{1}$$

where the inclusion map, j is $j\left(\lfloor g,X\rfloor\right) = \lfloor T_e(L_g)(X)\rfloor$ and the second map is $\pi_*\left(\lfloor X\rfloor\right) = T\pi(X)$. This is an exact sequence of vector bundles together with the suitable bracket structure induced on them, Theorem 18 and Corollary 20.

There are two alternative formulation of this sequence using of our results:

Firstly: The vector bundle isomorphism $I(M)\times\mathcal{I}(\mathcal{M}) \to T(I(M))$, which is $(g,X) \mapsto T_e(L_g)(X)$, respects the right actions of $(I(M))_m$. So quotients to a vector bundle isomorphism

$$\frac{I(M)\times\mathcal{I}(\mathcal{M})}{(I(M))_m} \to \frac{T(I(M))}{(I(M))_m},$$

where

$$\frac{I(M)\times\mathcal{I}(\mathcal{M})}{(I(M))_m}$$

is the bundle associated to

$$I(M)\left(\frac{I(M)}{(I(M))_m},(I(M))_m\right)$$

through the adjoint action of $(I(M))_m$ on $\mathcal{I}(\mathcal{M})$.

Likewise, there is a vector bundle isomorphism

$$\frac{I(M)\times\left(\frac{\mathcal{I}(\mathcal{M})}{(\mathcal{I}(\mathcal{M}))_m}\right)}{(I(M))_m} \to T\left(\frac{I(M)}{(I(M))_m}\right)$$

defined by the rule

$$\lfloor g, X + (\mathcal{I}(\mathcal{M}))_m\rfloor \mapsto T_e\left(\pi\circ L_g\right)(X),$$

where $(I(M))_m$ acts on the vector space

$$\frac{\mathcal{I}(\mathcal{M})}{(\mathcal{I}(\mathcal{M}))_m}$$

by $h\left(X + (\mathcal{I}(\mathcal{M}))_m\right) = Ad_hX + (\mathcal{I}(\mathcal{M}))_m$.

Thus the sequence of vector bundles (1) can be written as

$$\begin{array}{c} \dfrac{I(M)\times\left(\mathcal{I}(\mathcal{M})\right)_m}{(I(M))_m} \to \dfrac{I(M)\times\mathcal{I}(\mathcal{M})}{(I(M))_m} \\ \downarrow \\ \dfrac{I(M)\times\left(\dfrac{\mathcal{I}(\mathcal{M})}{\left(\mathcal{I}(\mathcal{M})\right)_m}\right)}{(I(M))_m} \end{array} \qquad (2)$$

where the inclusion map is $j_1\left(\lfloor g, X\rfloor\right)=\lfloor g, X\rfloor$ and the second is $f_1\left(\lfloor g, X\rfloor\right)=\left\lfloor g, X+\left(\mathcal{I}(\mathcal{M})\right)_m\right\rfloor$.

Secondly: The map

$$I(M)\times\mathcal{I}(\mathcal{M})\to\frac{I(M)}{(I(M))_m}\times\mathcal{I}(\mathcal{M}),$$

by the rule

$$(g, X)\mapsto\left(g\,(I(M))_m, Ad_g X\right),$$

which is a vector bundle morphism over

$$\pi: I(M)\to\frac{I(M)}{\left(I(M)\right)_m},$$

respects the action of

$$\left(I(M)\right)_m$$

on $I(M)\times\mathcal{I}(\mathcal{M})$ and so induces a vector bundle morphism

$$\frac{I(M)\times\mathcal{I}(\mathcal{M})}{(I(M))_m}\to\frac{I(M)}{(I(M))_m}\times\mathcal{I}(\mathcal{M}),$$

which is easily seen to be an isomorphism. So (1) can also be written as

$$\frac{I(M)\times\left(\mathcal{I}(\mathcal{M})\right)_m}{(I(M))_m}\to\frac{I(M)}{(I(M))_m}\times\mathcal{I}(\mathcal{M})\to T\left(\frac{I(M)}{(I(M))_m}\right) \qquad (3)$$

where the inclusion map, j_2 is

$$j_2\left(\lfloor g, X\rfloor\right)=\left(g(I(M))_m, Ad_g X\right)$$

and the second map, f_2 by the rule

$$f_2\left(g\,(I(M))_m, X\right)=T_e\left(\pi\circ R_{g^{-1}}\right)(X).$$

In this way, the central term in exact sequence associated to principal bundle $I(M)$, i.e.,

$$I(M)\left(\frac{I(M)}{(I(M))_m}\times(I(M))_m\right),$$

may be viewed as the bundle of infinitesimal displacement of the fibers.

V. CONCLUSION

In this paper, a new geometric approach was presented to calculate the number of time-frequency projections from some angles in filtered back-projection image which are equally spaced using of the surface area of the convex body and (symplectic) Holmes-Thompson volume. In fact, the calculations were made in terms of the volume of the Holmes-Thomson (as the bridge between Finsler geometry, integral geometry, and symplectic geometry). The projections from some angles which are equally spaced, the number of time-frequency projections from some angles which are equally spaced, and the low image correctness was corresponded to the notions of Finsler geometry. In this way, the proposed geometric approach can provide an appropriate answer to some medical imaging issues in optimum filtered back-projection images.

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