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# Free Vibration of Large Regular Repetitive Structures

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***Abstract-*** In this paper, free vibrations of large repetitive regular planar and space trusses are considered. For this kind of structures the corresponding stiffness and mass matrices have tri-diagonal block and in the canonical form, $\mathbf{K} = F(\mathbf{B}, \mathbf{A}, \mathbf{B}^T)$ and $\mathbf{M} = G(\mathbf{0}, \mathbf{D}, \mathbf{0})$, which simplifies their Eigen frequencies. Practical large space and planar trusses contains with repetitive units, known as their substructures, have structural matrices in the presented canonical form **K** and **M**. Here, an efficient method is presented for free vibration of this type of structures. Examples are included to show the accuracy of the presented approach.

***Keywords-*** *Free vibration, Eigen frequencies, tri-diagonal, Eigen solution, stiffness and mass matrix, canonical forms, truss, space structures*.

## I. INTRODUCTION

Many engineering problems require the calculation of the eigenvalues and eigenvectors of matrices. As an example, the eigenvalues correspond to natural frequencies in vibration systems and buckling loads in the stability analysis of structures [1–3]. In order to calculate the eigenvalues of a matrix, the characteristic equation of the matrix should be formed and the corresponding equation of order *n* should be solved. Solution of this equation for a large *n* is not only difficult but also it is often accompanied by some errors. In the past decade canonical forms are developed and used for Eigen solution of bi-lateral symmetric structures [4-5]. Other canonical forms consist of block tri-diagonal and block penta-diagonal matrices arising from more general symmetries and regular structures [6]. For tri-diagonal cases the corresponding matrices have often the form $\mathbf{M} = F(\mathbf{B}, \mathbf{A}, \mathbf{B})$, and the eigenvalues of these problems can be simplified using special decomposition methods [7,8]. In this paper, considering the properties of the matrices of the form $\mathbf{M} = F(\mathbf{B}, \mathbf{A}, \mathbf{B}^T)$, a special method is developed to simplify the calculations. This can be used in combinatorial optimization problems such as ordering and partitioning of graph models using Fiedler vector [9], and it can also be employed in stability and dynamic analyses of repetitive space structures and finite element models. In the present method, the structure is decomposed into repeated substructures and the solution for static analysis is obtained partially, and the problem of finding the Eigen frequencies of the main structures is transformed into calculating those of their special repeating substructures.

## II. FREE VIBRATION OF STRUCTURES

The equilibrium equations of motion for a freely vibrating for un-damped system can be in the form:

$$\mathbf{M}\ddot{U} + \mathbf{K}U = 0. \tag{1}$$

Here **M** is the mass and **K** is the stiffness matrix of the system. The problem of vibration analysis consists of determining the conditions under which the equilibrium condition expressed by Eq. (1) will be satisfied.
If we assume

$$U(t) = \overline{U}\sin(\omega t + \theta). \tag{2}$$

In this expression $\overline{U}$ represents the shape of the system (which does not change with time; only the amplitude varies) and $\theta$ is a phase angle. When the second time derivative of Eq. (2) is taken, the accelerations in free vibration are

$$\ddot{U}(t) = -\omega^2\overline{U}\sin(\omega t + \theta) = -\omega^2 U, \tag{3}$$

Substituting Esq. (2) and (3) into Eq. (1) gives

$$-\omega^2\mathbf{M}\overline{U}\sin(\omega t + \theta) + \overline{U}\mathbf{K}\sin(\omega t + \theta) = 0, \tag{4}$$

This (since the sine term is arbitrary and may be omitted) may be written:

$$(\mathbf{K} - \omega^2\mathbf{M})\overline{U} = 0. \tag{5}$$

Equation (5) is one way of expressing what is called an eigenvalue or characteristic value problem. The quantities $\omega^2$ are the eigenvalues or characteristic values indicating the square of the free vibration frequencies, while the corresponding displacement vectors $\bar{U}$ express the corresponding shapes of the vibrating system, Known as the eigenvectors or mode shapes. Hence a nontrivial solution is possible only when the denominator determinant vanishes. In other words, finite amplitude free vibrations are possible only when

$$\det(\mathbf{K}-\omega^2\mathbf{M})=0. \quad (6)$$

Equation (6) is called the frequency equation of the system. Expanding the determinant will give an algebraic equation of the $n$ th degree in the frequency parameter $\omega^2$ for a system having $n$ degrees of freedom. The $n$ roots of this equation ($\omega_1^2,\omega_2^2,\omega_3^2,...,\omega_n^2$) represent the frequencies of the N modes of vibration which are possible in the system.

## III. BLOCK TRI-DIAGONAL MATRICES AND THEIR SOLUTION FOR FREE VIBRATION OF TRUSSES

Consider the following block diagonal canonical form:

$$\mathbf{K}_{(n*m)(n*m)}=F(\mathbf{B}_{(m*m)},\mathbf{A}_{(m*m)},\mathbf{B}^T_{(mm)}), \quad (7)$$

$$\mathbf{M}_{(n*m)(n*m)}=F(\mathbf{0}_{(m*m)},\mathbf{C}_{(m*m)},\mathbf{0}_{(mm)}), \quad (8)$$

Where we have $n$ blocks on the diagonal as shown in Eq. (9) and $\mathbf{K}$ is the stiffness matrix of a planar or space truss and $\mathbf{M}$ is the lumped mass matrix. We assume n to be a large number.

$$\mathbf{K}=\begin{bmatrix}\mathbf{A}_{m*m} & \mathbf{B}^T_{m*m} & & & & \\ \mathbf{B}_{m*m} & \mathbf{A}_{m*m} & \mathbf{B}^T_{m*m} & & & \\ & & \cdots & & & \\ & & \mathbf{B}_{m*m} & \mathbf{A}_{m*m} & \mathbf{B}^T_{m*m} & \\ & & & \cdots & & \\ & & & \mathbf{B}_{m*m} & \mathbf{A}_{m*m} & \mathbf{B}^T_{m*m} \\ & & & & \mathbf{B}_{m*m} & \mathbf{A}_{m*m}\end{bmatrix}_{(n*m)(n*m)} \quad (9)$$

$$\mathbf{M}=\begin{bmatrix}\mathbf{C}_{m*m} & & & \\ & \mathbf{C}_{m*m} & & \\ & & \cdots & \\ & & & \mathbf{C}_{m*m}\end{bmatrix}_{(n*m)(n*m)} \quad (10)$$

For calculating the Eigen frequencies of system, the equation (11) should be solved in the following standard form:

$$\mathbf{K}\varphi=\lambda\mathbf{M}\varphi, \quad (11)$$

And in the developed form we have

$$\begin{bmatrix}\mathbf{A} & \mathbf{B}^T & & & & \\ \mathbf{B} & \mathbf{A} & \mathbf{B}^T & & & \\ \ddots & \ddots & \ddots & & & \\ & \mathbf{B} & \mathbf{A} & \mathbf{B}^T & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mathbf{B} & \mathbf{A} & \mathbf{B}^T \\ & & & & \mathbf{B} & \mathbf{A}\end{bmatrix}*\begin{bmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_i\\ \vdots\\ \varphi_{n-1}\\ \varphi_n\end{bmatrix}=\lambda\mathbf{M}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_i\\ \vdots\\ \varphi_{n-1}\\ \varphi_n\end{bmatrix}, \quad (12)$$

or

$$\begin{bmatrix}\mathbf{A}-\lambda\mathbf{C} & \mathbf{B}^T & & & & \\ \mathbf{B} & \mathbf{A}-\lambda\mathbf{C} & \mathbf{B}^T & & & \\ \ddots & \ddots & \ddots & & & \\ & \mathbf{B} & \mathbf{A}-\lambda\mathbf{C} & \mathbf{B}^T & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mathbf{B} & \mathbf{A}-\lambda\mathbf{C} & \mathbf{B}^T \\ & & & & \mathbf{B} & \mathbf{A}-\lambda\mathbf{C}\end{bmatrix}*\begin{bmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_i\\ \vdots\\ \varphi_{n-1}\\ \varphi_n\end{bmatrix}=0. \quad (13)$$

Expansion of the above matrix leads to:

$$\begin{cases}.... \\ ...... \\ \mathbf{B}\varphi_{i-2}+(\mathbf{A}-\lambda\mathbf{C})\varphi_{i-1}+\mathbf{B}^T\varphi_i=\mathbf{0} \quad \to i-1 \\ \mathbf{B}\varphi_{i-1}+(\mathbf{A}-\lambda\mathbf{C})\varphi_i+\mathbf{B}^T\varphi_{i+1}=\mathbf{0} \quad \to i \\ ....... \\ ....\end{cases} \quad (14)$$

Using the above set of equations we have:
From the ($i$–1)th row of Eq. (14) we consider

$$\frac{\varphi_{i-2}}{\varphi_{i-1}}=\alpha, \quad \frac{\varphi_{i-1}}{\varphi_i}=\beta. \quad (15)$$

and from the $i$th row of Eq. (14) we define:

$$\frac{\varphi_{i-1}}{\varphi_i}=\gamma, \quad \frac{\varphi_i}{\varphi_{i+1}}=\eta. \quad (16)$$

The values of $\alpha,\beta,\gamma,\eta$ can easily be formed, and since the matrix is considered to have high dimension, therefore we can accept the following approximation:

$$\alpha\cong\gamma\cong\beta\cong\eta, \quad (17)$$

Suppose

$$\alpha\cong\gamma=\beta\cong\eta\cong e^{i\theta}. \quad (18)$$

Considering Eq. (17) and Eq. (18) we have

$$\frac{\varphi_{i-1}}{\varphi_i}=e^{i\theta}, \quad \frac{\varphi_i}{\varphi_{i+1}}=e^{i\theta}, \quad (19)$$

$$\varphi_{i-1} = e^{i\theta}\varphi_i, \quad \varphi_{i+1} = e^{-i\theta}\varphi_i. \tag{20}$$

Substituting Eq. (18) in the $i$ th row of Eq. (12) leads to

$$e^{i\theta}\mathbf{B}\varphi_i + (\mathbf{A} - \lambda\mathbf{C})\varphi_i + e^{-i\theta}\mathbf{B}^T\varphi_i = 0 \tag{21}$$
$$(e^{i\theta}\mathbf{B} + \mathbf{A} + e^{-i\theta}\mathbf{B}^T)\varphi_i = \lambda\mathbf{C}\varphi_i. \tag{22}$$

Equation (22) shows that the Eigen frequencies of the matrix **M** can be obtained from the eigenvalues of $(e^{i\theta}\mathbf{B} + \mathbf{A} + e^{-i\theta}\mathbf{B}^T)$ and **C**, (i.e. in terms of $n$ diagonal blocks). Using $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ simplifies Eq. (22) as

$$\bigcup_{j=1}^{n} \det[\mathbf{A} + \cos(\theta) * (\mathbf{B} + \mathbf{B}^T) + i * \sin(\theta) * (\mathbf{B} - \mathbf{B}^T) - \lambda\mathbf{C}] = \mathbf{0}, \quad \theta = \frac{j * \pi}{n+1}. \tag{23}$$

Using the above relationship the solution of large matrices becomes possible using the properties of its small blocks.

## IV. NUMERICAL EXAMPLES

**Example 1**: As a first example consider the following practical planar truss which showed in Figure 1.

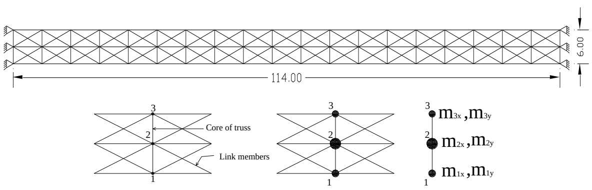


Figure 1. Planar truss and its cores, link members and assigned masses

In this example modules of elasticity of the truss elements is $E = 2*10^6 \frac{kg}{cm^2}$ and area of elements A=76 $cm^2$. The stiffness and mass matrices of planar truss can be written as

$$\mathbf{K} = \begin{bmatrix} \mathbf{A}_{6*6} & \mathbf{B}_{6*6} & & \\ \mathbf{B}^T_{6*6} & \mathbf{A}_{6*6} & \mathbf{B}_{6*6} & \\ & \ddots & \ddots & \mathbf{B}_{6*6} \\ & & \mathbf{B}^T_{6*6} & \mathbf{A}_{6*6} \end{bmatrix}_{108*108}, \mathbf{M} = \begin{bmatrix} \mathbf{C}_{6*6} & & & \\ & \mathbf{C}_{6*6} & & \\ & & \ddots & \\ & & & \mathbf{C}_{6*6} \end{bmatrix}_{108*108},$$

$$\mathbf{C} = \begin{bmatrix} m_{1x} & & & & & \\ & m_{1y} & & & & \\ & & m_{2x} & & & \\ & & & m_{2y} & & \\ & & & & m_{3x} & \\ & & & & & m_{3y} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 8692087 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5973020 & 0 & -5066667 & 0 & 0 \\ 0 & 0 & 12317490 & 0 & 0 & 0 \\ 0 & -5066667 & 0 & 11946039 & 0 & -5066667 \\ 0 & 0 & 0 & 0 & 8692078 & 0 \\ 0 & 0 & 0 & -5066667 & 0 & 5973020 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} -253333 & 0 & -1812706 & 906352.9 & 0 & 0 \\ 0 & 0 & 906352.9 & -453176 & 0 & 0 \\ -1812706 & -906353 & -2533333 & 0 & -1812706 & 906352.9 \\ -906353 & -453176 & 0 & 0 & 906352.9 & -453176 \\ 0 & 0 & -1812706 & -906353 & -2533333 & 0 \\ 0 & 0 & -906353 & -453176 & 0 & 0 \end{bmatrix}.$$

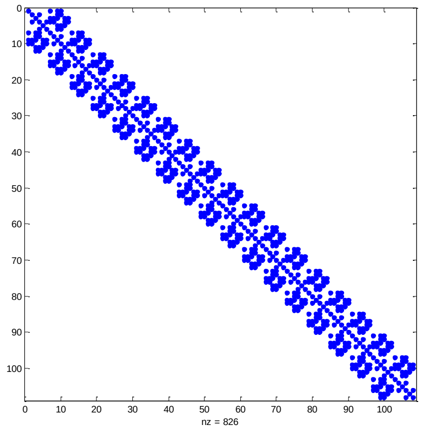


Figure 1 Pattern of stiffness matrix of truss

Assigned mass for each joint in x and y directions are $m_{1x} = m_{3x} = 1000, m_{2x} = 2000$ and $m_{1y} = m_{3y} = 2000, m_{2y} = 4000 \frac{kg_f.\sec^2}{m}$.

Eigenfrequencies calculated by exact and present method and compared it table (1) and figure (2) respectively.

Table (1), comparison of 10 last frequencies of truss

| **Ten last frequencies of truss** | |
|---|---|
| ***Exact method*** | ***Present method*** |
| 55.836 | 55.841 |
| 56.831 | 56.846 |
| 57.786 | 57.794 |
| 57.955 | 57.958 |
| 58.477 | 58.480 |
| 58.895 | 58.896 |
| 59.728 | 59.730 |
| 61.130 | 61.131 |
| 62.142 | 62.142 |
| 62.753 | 62.754 |

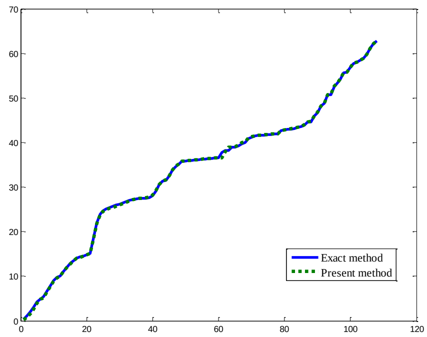


Figure 2. Comparison of the frequencies of the truss using the present method with 18 repeated substructure, and the exact approach

**Example 2**: As another practical example, consider the space structures illustrated in Figure 3.

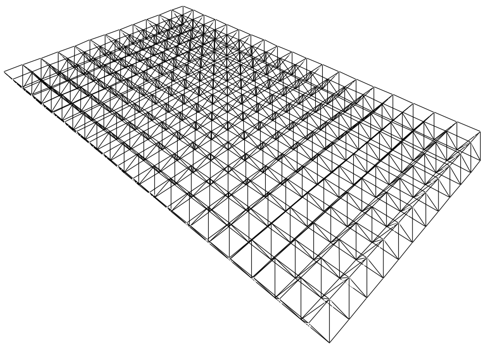

Figure 3. A space structures

In this example also modules of elasticity of the truss elements is $E = 2*10^6 \frac{kg}{cm^2}$ and area of elements A=76 $cm^2$. The stiffness and mass matrices of planar truss can be written as

$$\mathbf{K} = \begin{bmatrix} \mathbf{A}_{63*63} & \mathbf{B}_{63*6} & & \\ \mathbf{B}^T_{63*63} & \mathbf{A}_{63*63} & \mathbf{B}_{63*63} & \\ & \cdots & & \mathbf{B}_{63*63} \\ & & \mathbf{B}^T_{63*63} & \mathbf{A}_{63*63} \end{bmatrix}_{1134*1134},$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{C}_{63*63} & & \\ & \mathbf{C}_{63*63} & \\ & & \ddots & \\ & & & \mathbf{C}_{63*63} \end{bmatrix}_{1134*1134}, \mathbf{C} = \begin{bmatrix} 1000 & & & & & \\ & 1000 & & & & \\ & & 1000 & & & \\ & & & \cdots & & \\ & & & & 1000 & \\ & & & & & 1000 \end{bmatrix}_{63*63}.$$

The pattern of the stiffness matrix for space truss shown in Figure 3 is illustrated in Figure 4.

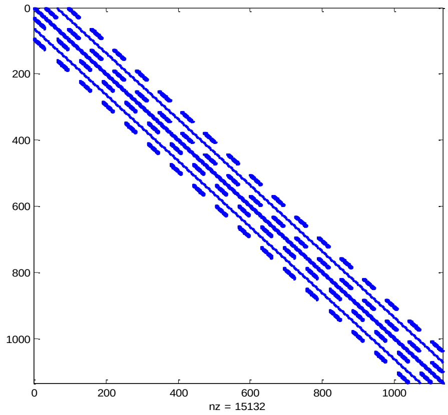


Figure 4. Pattern of stiffness matrix of the space structures

In this example for each joint considered mass are

$$m_x = m_y = m_z = 1000 \frac{kg_f . \sec^2}{m}.$$

Eigenfrequences calculated by exact and present method and compared in table (2) and figure (5) respectively.

Table (2) Comparison of frequencies of truss

| **Comparison of frequencies** | | | | | |
|---|---|---|---|---|---|
| ***Ten first frequencies*** | | ***Ten middle frequencies*** | | ***Ten last frequencies*** | |
| Exact method | present method | Exact method | present method | Exact method | present method |
| 3.9947 | 3.7742 | 74.8220 | 74.9324 | 112.4900 | 112.4923 |
| 5.3044 | 4.5196 | 74.8850 | 74.9906 | 112.5500 | 112.5173 |
| 7.7962 | 6.5974 | 75.1050 | 75.0236 | 112.5700 | 112.5768 |
| 8.9903 | 8.8962 | 75.1510 | 75.1257 | 112.5900 | 112.5823 |
| 9.2318 | 9.1731 | 75.1770 | 75.1339 | 112.6200 | 112.7394 |
| 9.6131 | 9.2067 | 75.2340 | 75.2526 | 112.7500 | 112.7655 |
| 11.0540 | 9.7191 | 75.2390 | 75.2560 | 113.1800 | 113.1847 |
| 11.1190 | 10.3477 | 75.2960 | 75.3791 | 113.2600 | 113.2597 |
| 11.3680 | 11.2229 | 75.3150 | 75.4000 | 113.6600 | 113.6633 |
| 13.6180 | 12.5265 | 75.3470 | 75.4020 | 113.7300 | 113.7326 |

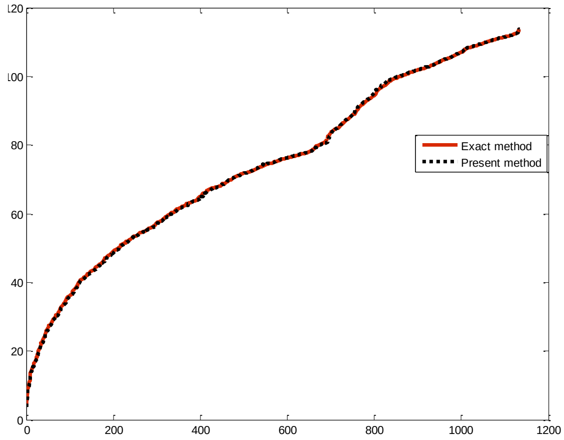


Figure 5. Comparison of the frequencies of the truss using the present method and the exact approach

## V. CONCLUDIND REMARKS

In this paper, a simple method is presented for calculating the Eigen-frequencies of large structural matrices of the form $\mathbf{M} = F(\mathbf{B}, \mathbf{A}, \mathbf{B}^T)$. Examples are solved and the results are compared to exact solutions. The eigenvalues are very close to the exact values, and can efficiently be used for solution of the models whose structural matrices are or can be transformed into the block tri-diagonal form $\mathbf{M} = F(\mathbf{B}, \mathbf{A}, \mathbf{B}^T)$. The application is by no means limited to dynamic analyses of different types of repetitive structures. This method can also be considered as powerful tool for iterative methods. By using this method one can does shape approximate a structure and use its result for initial values in iterative methods for improved results. The present method can also be used for pre-conditioning of matrices for fast convergence of in iterative methods.

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