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Maximum Entropy and MAP Estimation Using Conjugate Gradient Method for Phase Unwrapping

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Abstract- We constructed a practical and useful method for phase unwrapping in remote sensing using synthetic aperture radar (SAR) interferometry. First, we constructed a method of maximum entropy to achieve phase unwrapping with high degree of accuracy. Using Monte Carlo simulation for one dimensional artificial wave-front, we found that the method of maximum entropy served an accurate method for phase unwrapping, if we assumed an appropriate model of true prior. Then, in order to construct a practical method, we constructed a deterministic limit of the method of maximum entropy based on a maximum of a posteriori (MAP) estimation using conjugate gradient (CG) method. Using numerical simulation for artificial wave-fronts, we found that the CG method realized phase unwrapping with high degree of accuracy using an appropriate model of true prior. Also, we found that the CG method realized phase unwrapping accurately by utilizing sets of unwrapped phase differences composed from second differences of the observed wave-fronts, even if aliasing occurred at several sampling points.

Keywords- *Bayesian inference; MAP estimation; phase unwrapping; conjugate gradient method.*

I. INTRODUCTION

For many years, researchers have investigated information communication utilizing wave-fronts, such as remote sensing due to the synthetic aperture radar (SAR) interferometry. In such field, a lot of engineers have constructed optical measurement systems to observe an interferogram by utilizing interferometer. Then, a technique called as phase unwrapping is essential to derive information on wave-fronts from the interferogram. For this problem, researchers have constructed various techniques [1]-[3] for phase unwrapping, such as least square estimation and its variants [4]-[7], Bayesian inference [8], [9] using simulated annealing and a method of maximum entropy. In recent years, Saika and Nishimori [10], [11], Saika and Uezu [12] have investigated phase unwrapping based on an analogy between statistical mechanics and Bayesian inference using the maximizer of the posterior marginal (MPM) estimate. Then, Marroquin and Rivera [13] have constructed the MAP estimation using conjugate gradient (CG) method for phase unwrapping in remote sensing using SAR interferometry. They found that maximum of a posteriori (MAP) estimation using the CGM succeeded in phase unwrapping under several conditions. Sakaematsu and Saika [14] have improved performance of the MAP estimation using the CGM for two dimensional phase unwrapping. However, as they have not tried a systematic approach for this problem, it was not clarified criterion that the CGM based on the MAP estimation was effective for phase unwrapping in remote sensing using SAR interferometry.

Therefore, in this study, we tried two kinds of Bayesian modeling for one and two dimensional phase unwrapping in remote sensing using SAR interferometry. First, we investigated the method of maximum entropy for phase unwrapping in remote sensing using the SAR interferometry to estimate efficiency of probabilistic inference for this problem. Also, we tried the MAP estimation using the CG method to construct a practical method for phase unwrapping in remote sensing using SAR interferometry. Here, we used the term “the MAP estimation” to represent a deterministic limit of a method of the maximum entropy. In the case of the method of maximum entropy, we carried out phase unwrapping so as to maximize the information entropy. Using Monte Carlo simulation for several wave-fronts in one dimensions, we found that the present method was successful in phase unwrapping with high degree of accuracy, if we assumed a model of true prior appropriately. From this fact, we clarified that the probabilistic inference was effective for phase unwrapping with a use of appropriate model of true prior and likelihood approximating noise probability. Also, we investigated the MAP estimation using the CG method for one and two dimensional phase unwrapping in remote sensing using the SAR interferometry. In our method, we first composed a set of unwrapped phase differences by using the CG method from sets of differences of phase differences in a principal interval [$-\pi$, π]. Then, we reconstructed the original wave-front using the set of unwrapped phase differences. Here, we estimated performance for interferograms observed from artificial wave-fronts. If aliasing did not occurred at every sampling point, numerical simulations clarified that the MAP estimation using the CG method carried out phase unwrapping perfectly without using prior information, if phase differences were not corrupted by any noises. Also, we found that the CG method succeeded in noise reduction using an appropriate model of the true prior, even if the original wave-fronts were corrupted by additive white Gaussian noises. Next, if aliasing occurred at several sampling points, numerical simulations clarified that the MAP estimation using the CG method was

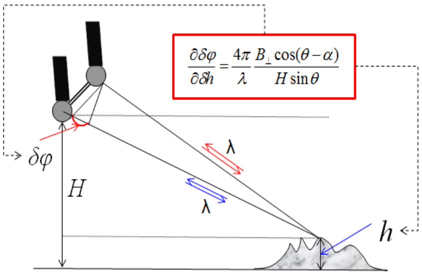

Fig. 1 Remote sensing using SAR interferometry

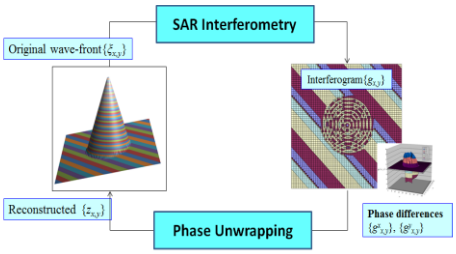

Fig. 2 Optical measurement via SAR interferometry and phase unwrapping

successful in phase unwrapping introducing a pre-procedure in which the CG method reconstructs two sets of unwrapped phase differences from four sets of differences of phase differences in the principal interval. We first confirmed that the present method realized phase unwrapping perfectly without using prior information, if original wave-front was not corrupted by any noises. Also, we found that the CG method succeeded in noise reduction on original wave-fronts with high degree of accurately, if we adjusted parameters appropriately.

The content of this paper is as follows. First, we outlined general formulation for phase unwrapping both in one and two dimensions. Then, we investigated performance for phase unwrapping using numerical simulations for wave-fronts in remote sensing via SAR interferometry. Last part is devoted to summary and discussion.

II. General Formulation

Here, we outlined Bayesian probabilistic formulation for phase unwrapping in remote sensing using SAR interferometry (Figs. 1 and 2). Here we showed two methods. One was based on the method of maximum entropy and the other was based on the MAP estimation using the CG method.

In this formulation, we first consider an original wave-front $\{\xi_i\}/\{\xi_{x,y}\}$ both in one and two dimensions. Here, $0<\xi_i\,(\xi_{x,y})<R_0$, $i=1,\ldots,L$ (x, $y=1,\ldots,L$). As shown in Figs. 3(a) and (e), we use one dimensional artificial wave-fronts in Figs. 3(a) and (e) and also use two dimensional one (Fig. 4 (a)) on the square lattice in Fig. 5(a). Next, when the original wave-front $\{\xi_i\}/\{\xi_{x,y}\}$ is carried through a noisy transmission, the original wave-front $\{\xi_i\}/\{\xi_{x,y}\}$ is corrupted by some noises, such as the atmospheric disturbance. Here, we consider the additive white Gaussian noise as

$$\eta_i=\xi_i+n_i(0,\sigma^2) \tag{1}$$

in one dimension and

$$\eta_{x,y}=\xi_{x,y}+n_{x,y}(0,\sigma^2) \tag{2}$$

in two dimensions. In this study, $n_i(0,\sigma^2)/n_{x,y}(0,\sigma^2)$ is the additive white Gaussian noise. Here, we set to σ=0, 0.2, 0.4. Next, we observe one dimensional interferogram:

$$\zeta_i=\mathrm{mod}(\eta_i+\pi,2\pi)-\pi \tag{3}$$

in Fig. 3(b) and two dimensional one:

$$\zeta_{x,y}=\mathrm{mod}(\eta_{x,y}+\pi,2\pi)-\pi \tag{4}$$

in Fig. 4(b). Then, we derive a set of phase differences $\{g^x{}_i\}/\{g^x{}_{x,y}\}(\{g^y{}_{x,y}\})$ in Figs. 3(c) and Fig. 4(c) from the interferograms in eqs. (3) and (4) as

$$g^x{}_i=\mathrm{mod}(\eta_{i+1}-\eta_i+\pi,2\pi)-\pi \tag{5}$$

in one dimension and

$$g^x{}_{x,y}=\mathrm{mod}(\eta_{x+1,y}-\eta_{x,y}+\pi,2\pi)-\pi \tag{6}$$

$$g^y{}_{x,y}=\mathrm{mod}(\eta_{x,y+1}-\eta_{x,y}+\pi,2\pi)-\pi \tag{7}$$

in two dimensions. Here, Fig. 5(a) shows relation between sampling points of interferometer and lattice points of the original wave-front. In this study, we assume that phase differences are not corrupted by any noises, when we observe them due to the optical instruments via the interferometer.

Next, we carry out phase unwrapping with the use of information on interferograms, such as phase differences $\{g^x{}_i\}/\{g^x{}_{x,y}\}(\{g^y{}_{x,y}\})$. In this study, we utilize two kinds of Bayesian inference using the method of maximum entropy and the MAP estimation using the CG method. For our purpose, we use a model system $\{z_i\}/\{z_{x,y}\}$ ($0<z_i/z_{x,y}<\infty$, $i=1,\ldots,L/$ x, $y=1,\ldots,L$). In two dimensional case, we use the model system $\{z_{x,y}\}$ on the square lattice in Fig. 5(b). When we carry out the method of maximum entropy, we reconstruct original wave-front as an expectation $z_i/z_{x,y}$ averaged over probability distribution:

$$\Pr(\{z_i\})\propto\exp\left[-\frac{1}{T_m}E(\{z_i\})\right], \tag{8}$$

where

$$E(\{z_i\})=\sum_i(z_{i+1}-z_i-g_i^x)^2+\lambda\sum_i(z_{i+1}-2z_i+z_{i-1})^2 \tag{9}$$

in one dimension under a free boundary condition. Then, in two dimensions, we utilize

$$\Pr(\{z_{x,y}\})\propto\exp\left[-\frac{1}{T_m}E(\{z_{x,y}\})\right], \tag{10}$$

where

$$\begin{aligned}E(\{z_{x,y}\})&=\sum_{(x,y)}(z_{x+1,y}-z_{x,y}-g_{x,y}^x)^2+\sum_{(x,y)}(z_{x,y+1}-z_{x,y}-g_{x,y}^y)^2\\&+\lambda\sum_{(x,y)}(z_{x+1,y}-2z_{x,y}+z_{x-1,y})^2+\lambda\sum_{(x,y)}(z_{x,y+1}-2z_{x,y}+z_{x,y-1})^2\\&+\lambda\sum_{(x,y)}(z_{x+1,y+1}-z_{x,y+1}-z_{x+1,y}+z_{x,y})^2\end{aligned} \tag{11}$$

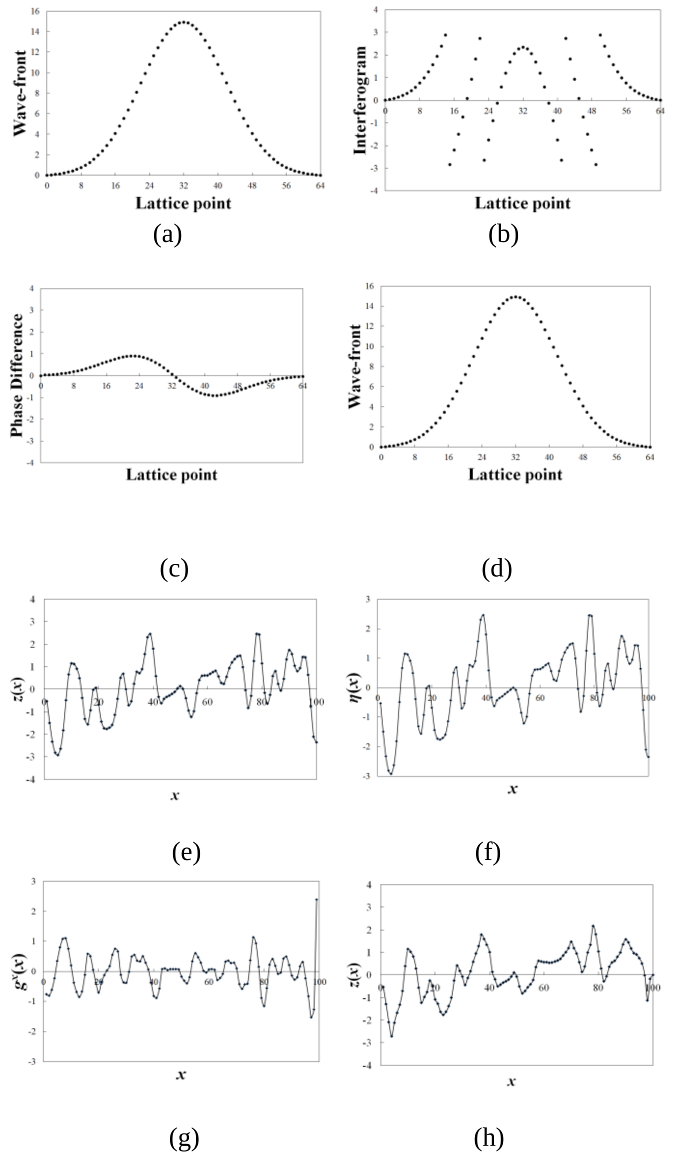

Fig. 3 (a) an original wave-front in remote sensing using the SAR interferometry, (b) an interferogram from the wave-front (a), (c) principal values of phase differences obtained from the interferogram (b), (d) a reconstructed wave-front due to the MAP estimation using the CGM under the optimal condition, (e) a wave-front generated with probability distribution expressed as a Boltzmann factor of the Ising model in one dimension, (f) a reconstructed wave-front by the method of maximum entropy for the original wave-front (e).

under a free boundary condition. In order to carry out the method of maximum entropy, we utilize the Monte Carlo simulation based on the Metropolis algorithm. Here, we examine efficiency of fluctuations around the MAP solution by making use of the method of maximum entropy.

On the other hand, we try the CG method based on the MAP estimation which is regarded as a Tm→0 limit of the method of maximum entropy, i. e.,

$$z_i = \lim_{T_m \to 0} \langle z_i \rangle_{T_m} \tag{12}$$

$$\langle z_i \rangle_{T_m} = \frac{1}{Z} \prod_{i=1}^{L} \left(\int dz_i \right) \exp\left[-\frac{1}{T_m} E(\{z_i\}) \right] z_i \tag{13}$$

Here, we apply the CGM to obtain the MAP solution by using the minimum condition of the cost function in eq. (9) as

$$\frac{\partial E(\{z_i\})}{\partial z_i} = 0. \tag{14}$$

at each lattice point. Then, in the two dimensional case, we utilize the CGM to reconstruct the original wave-front as

$$z_{x,y} = \lim_{T_m \to 0} \langle z_{x,y} \rangle_{T_m} \tag{15}$$

$$\langle z_{x,y} \rangle_{T_m} = \frac{1}{Z} \prod_{(x,y)}^{L^2} \left(\int dz_i \right) \exp\left[-\frac{1}{T_m} E(\{z_{x,y}\}) \right] z_{x,y} \tag{16}$$

using the model system $\{z_{x,y}\}(0<z_{x,y},<\infty,\ x,\ y=1,\ldots,L)$ on the square lattice. In this case, we carry out the CGM to obtain the MAP solution based on the minimum condition of the cost functions in eq. (11) as

$$\frac{\partial E(\{z_{x,y}\})}{\partial z_{x,y}} = 0. \tag{17}$$

at each lattice point on the square lattice. Here, in convenience, we note that a set of linear equations on $\{z_i\}/\{z_{x,y}\}$ as

$$Az = b, \tag{18}$$

which represents the minimum conditions of the cost function in eqs. (9) and (11). Here, z is a L-dimensional vector which expresses a wave-front and b is a L-dimensional constant vector. Then, A is a $L\times L/L^2\times L^2$ matrix. Next, we indicate how to obtain the MAP solution via the CGM as below.

Algorithm of the CGM
(i) First, we set to $\boldsymbol{z}_0 = 0$, $\boldsymbol{r}_0 = \boldsymbol{b}_0 - A\boldsymbol{z}_0$, $\boldsymbol{p}_0=\boldsymbol{r}_0$,
(ii) Then, we set to k=0.
(iii) Next, we calculate
$\alpha_k = \{r_k, p_k\}/\{p_k, Ap_k\}$,
where $\{a, b\}=\sum_i a_i b_i$.
(iv) We set to $z_{k+1} = z_k+\alpha_k p_k$ and $r_{k+1} = r_k - \alpha_k A p_k$.
(v) If $\|r_{k+1}\|<\varepsilon$, stop. Otherwise, we calculate
$\beta_k = -\{r_{k+1}, r_{k+1}\}/\{r_k, r_k\}$,
$p_{k+1} = r_{k+1} + \beta_k p_k$.
and set to k=k+1, then go to (iii).

In addition, we then treat a case that aliasing occurs in optical measurements using interferometer. In this method, we first construct a method which make a set of unwrapped phase differences by using sets of differences of the wrapped phase differences, such as $\{g^{xx}_i\}$ in one dimension and $\{g^{xx}_{x,y}\}$, $\{g^{xy}_{x,y}\}$, $\{g^{yx}_{x,y}\}$ and $\{g^{yy}_{x,y}\}$ in two dimensions. For this purpose, we carry out the CG method based on the MAP estimation to search the MAP solution of the assumed cost function:

$$E(\{z_x\}) = \sum_x (z_{x+1} - z_x - g^{xx}_x)^2 + \lambda \sum_x (z_{x+1} - 2z_x + z_{x-1})^2 \tag{19}$$

in one dimension. Here, we assumed that the difference of wave-front has a property to enhance smooth structures. Then, in the two dimensional case, we reconstruct the set of unwrapped version of the phase differences $\{g^{x}_{x,y}\}$ on the basis of the MAP estimation using the CGM via the cost function as

$$
\begin{aligned}
&E(\{z_{x,y}\}) \\
&= \sum_{(x,y)} (z_{x+1,y} - z_{x,y} - g^{xx}_{x,y})^2 + \sum_{(x,y)} (z_{x,y+1} - z_{x,y} - g^{xy}_{x,y})^2 \\
&+ \lambda \sum_{(x,y)} (z_{x+1,y} - 2z_{x,y} + z_{x-1,y})^2 + \lambda \sum_{(x,y)} (z_{x,y+1} - 2z_{x,y} + z_{x,y-1})^2 \qquad (20) \\
&+ \lambda \sum_{(x,y)} (z_{x+1,y+1} - z_{x,y+1} - z_{x+1,y} + z_{x,y})^2
\end{aligned}
$$

using the sets of second differences of the wave-fronts, $\{g^{xx}_{x,y}\}$ and $\{g^{xy}_{x,y}\}$. We here use this model so as to enhance smooth structures, as seen from a pattern of unwrapped phase differences. Also, we reconstruct the unwrapped version of phase differences $\{g^{y}_{x,y}\}$ by making use of the cost function as

$$
\begin{aligned}
&E(\{z_{x,y}\}) \\
&= \sum_{(x,y)} (z_{x+1,y} - z_{x,y} - g^{yx}_{x,y})^2 + \sum_{(x,y)} (z_{x,y+1} - z_{x,y} - g^{yy}_{x,y})^2 \\
&+ \lambda \sum_{(x,y)} (z_{x+1,y} - 2z_{x,y} + z_{x-1,y})^2 + \lambda \sum_{(x,y)} (z_{x,y+1} - 2z_{x,y} + z_{x,y-1})^2 \qquad (21) \\
&+ \lambda \sum_{(x,y)} (z_{x+1,y+1} - z_{x,y+1} - z_{x+1,y} + z_{x,y})^2
\end{aligned}
$$

using the sets of second differences of the wave-fronts, $\{g^{yx}_{x,y}\}$ and $\{g^{yy}_{x,y}\}$.

Next, in order to clarify efficiency of the present method, we evaluate the performance measure based on the mean square error as

$$\sigma = \frac{1}{L^2} \sum_{x=1}^{L} \sum_{y=1}^{L} \left(z_{x,y} - \xi_{x,y} \right)^2 . \qquad (22)$$

If this value becomes zero, if the phase unwrapping is carried out completely.

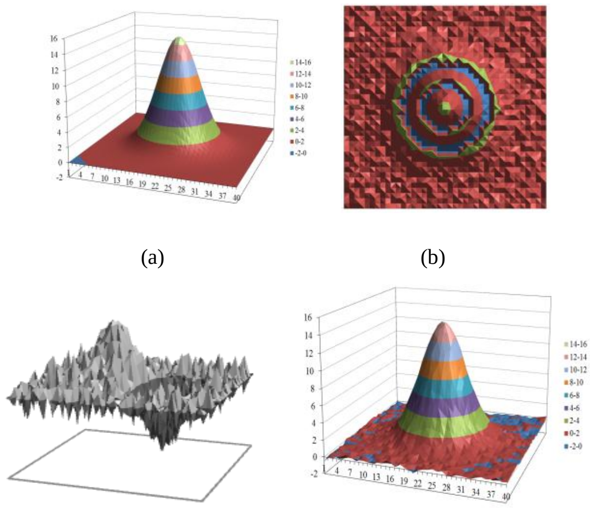

Fig. 4 (a) an original wave-front typical in remote sensing via SAR interferometry, (b) an interferogram obtained from the original wave-front in (a), (c) a set of phase differences obtained from the interferogram in (b), (d) a reconstructed wave-front by the MAP estimation using the CGM under the optimal condition.

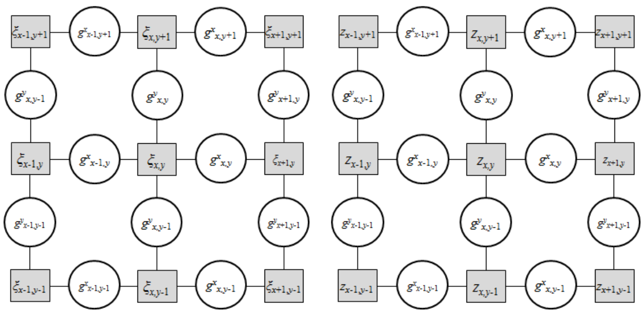

Fig. 5 (a) Lattice point (x,y) of the original wave-front $\{\xi_{x,y}\}$ and sampling points of phase differences $\{g^{x}_{x,y}\}$ and $\{g^{y}_{x,y}\}$ in two dimensions, (b) lattice point (x,y) of the original wave-front $\{z_{x,y}\}$ and sampling points of the phase differences $\{g^{x}_{x,y}\}$ and $\{g^{y}_{x,y}\}$ in two dimensions.

III. PERFORMANCE

In this study, we investigated statistical performance of the Bayesian inference using the method of maximum entropy and the MAP estimation for phase unwrapping in remote sensing using the SAR interferometry.

First, we estimated performance of these methods both for one and two dimensional phase unwrapping for the wave-fronts (Figs. 3(a), (e) and Fig. 4(a)). As seen from the interferograms in Figs. 3(b), (f) and Fig. 4(b), aliasing did not occurred at every sampling point. Here, we obtained the sets of the phase differences (Figs. 3 (c), (g) and Fig. 4(c)). Next, when we utilized the method of maximum entropy for phase unwrapping, we carried out the Monte Carlo simulation based on the Metropolis algorithm with 5000 Monte Carlo steps (MCS). As shown in Figs. 3(d), (h), we found that the present method succeeded in phase unwrapping with high degree of accuracy without using prior information, if wave-fronts were not corrupted by any noises. Also, we found that the method succeeded in reducing noises, if we assumed the model of the true prior appropriately. From these facts, we indicated that the Bayesian inference worked effectively for one dimensional phase unwrapping, if we assumed an appropriate model of the true prior which enhanced smooth structures, as seen from in realistic wave-fronts in SAR interferometry.

Also, we studied performance of the MAP estimation via the CG method, if there was no residue in interferogram of the wave-front in Fig. 4(a). As shown in Fig. 6, we evaluated λ dependence of the MSE averaged over 5 sets of the phase differences corrupted from the original wave-front by the Gaussian noise $N(0,\sigma^2)$, where σ=0, 0.4. We found that the CG method perfectly carried out phase unwrapping without using prior information, although this model is not so effective due to the additive white Gaussian noises on the phase differences.

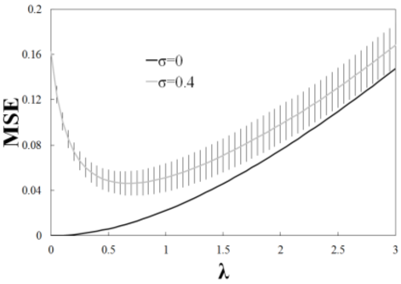

Fig. 6 Mean square error as a function of *λ*, if aliasing does not occur in optical measurements if σ=0, 0.4 and α= 1 in two dimensions.

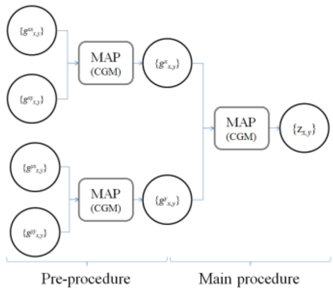

Fig. 7 Model system used for the MAP estimation using the CGM for the case that aliasing occurs at several sampling points.

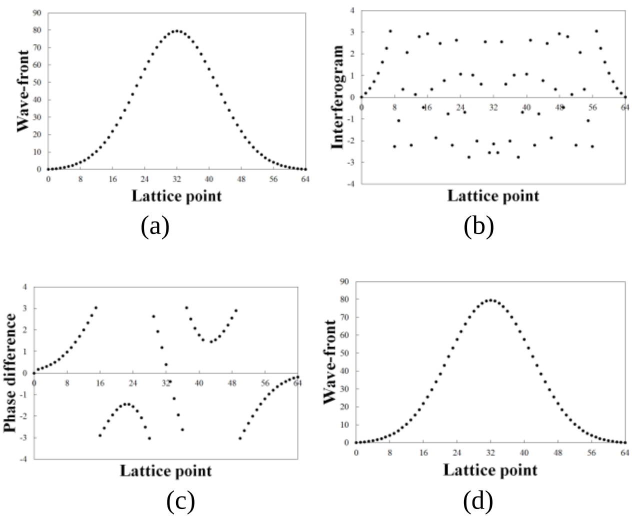

Fig. 8 (a) an original wave-front in one dimension, (b) an interferogram of the original wave-front (a), (c) phase differences in principal interval, (d) a reconstructed wave- front obtained by the MAP estimation via the CGM.

Also, we clarified that the CG method reconstructs the original wave-front with high degree of accuracy, if we assume the model of the prior information appropriately (λ=0.6(=λ_{OPT})), and however that the CG method is not effective due to the over-smoothing, if we set to $\lambda >> \lambda_{OPT}$. These results indicated that assuming an appropriate model prior was important to realize phase unwrapping with high degree of accuracy.

Next, we studied the MAP estimation using the CG method based on the framework in Fig. 7 for the case that aliasing occurred at several sampling points, as shown in Fig. 8(a) and Fig. 9(a). Here, as shown in Figs. 8(c), we found that the discontinuity ~2π appears in phase differences in principal interval [-π, +π] from the pattern of the interferogram in Fig. 8(b). In this method, we first used 4 sets of differences of wrapped phase differences, $\{g^{xx}_{x,y}\}$, $\{g^{xy}_{x,y}\}$, $\{g^{yx}_{x,y}\}$, $\{g^{yy}_{x,y}\}$ to construct unwrapped phase differences $\{g^{x}_{x,y}\}$, $\{g^{y}_{x,y}\}$. Then, utilizing these unwrapped phase differences, we carried out phase unwrapping using the CG method. When we estimated performance, we utilized the MSE which was averaged over 5 interferograms observed by interferometer. Here, as shown in Fig. 10, we first evaluated λ dependence of the MSE averaged over 5 sets of the phase differences (Fig. 9(c)) from the original wave-front (Fig. 9 (a)). First, we confirmed that the original wave-front was perfectly reconstructed by using the MAP estimation using the CG method without using prior information on original wave-fronts and its wave-front slopes. Then, we evaluated λ dependence of the MSE averaged over 5 sets of phase differences, if wave-fronts were corrupted by the Gaussian noise (σ=0, 0.2). First, we confirmed that the MAP estimation using the CG method carried out phase unwrapping perfectly without using prior information on wave-fronts and their wave-front slopes, i.e., λ_{PRE}=0 and λ=0. Then, as shown in Fig. 10, we found that the accuracy of reconstructed wave-front was suppressed by introducing the Gaussian noise N(0,σ=0.2) into the phase differences, if we did not utilize prior information. However, as shown in Fig. 10, we found that the optimal performance was realized at $\lambda=\lambda_{OPT}$ (>0) which was respective of choice of the parameter λ_{PRE}.

These results suggested that the MAP estimation via the CGM may be a practical method for phase unwrapping by making use of techniques of parameter-estimation, such as the EM algorithm.

IV. Summary and Discussion

In previous sections, we have constructed a practical method for phase unwrapping in remote sensing using the SAR interferometry by utilizing two methods, such as the method of maximum entropy and the MAP estimation using the CG method with the pre-procedure which constructed unwrapped phase differences from differences of phase differences in principal interval. Then, in order to clarify both the efficiency of fluctuations around the MAP estimation and the applicability of the CG method to phase unwrapping, we have investigated performance of these methods for several models approximating wave-fronts in remote sensing using the SAR interferometry. For our purpose, we have carried out numerical simulations for this problem, such as the Monte Carlo simulation and the CG method to solve set of linear equations. First, using the Monte Carlo simulation for the one-dimensional wave-fronts, we have found that the fluctuations around the MAP solution were available of accurate phase unwrapping, if we set to the parameter T_m appropriately. Then, in order to clarify the availability of the MAP estimation using the CG method, we have carried out numerical simulations both for one and two dimensional models representing wave-fronts in remote sensing using the SAR interferometry. We have found that phase unwrapping was carried out perfectly, if the original wave-fronts were not corrupted by any noises …

points, by introducing the pre-procedure which constructed unwrapped phase differences from sets of differences of phase differences derived from observed interferograms.

These results have suggested that the MAP estimation using the CG method may be a practical method for phase unwrapping introducing a technique of parameter estimation, such as the EM algorithm. As a future problem, we are going to construct a method for phase unwrapping which can be applicable of realistic case, such as remote sensing using the SAR interferometry.

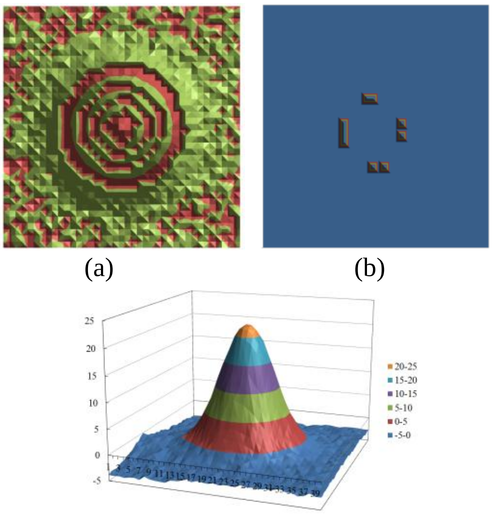

Fig. 9 (a) an interferogram of an original wave-front, (b) a pattern of residues of the interferogram (a), (c) a reconstructed wave-front under the optimal condition obtained by the MAP estimation using the CGM using the set of unwrapped phase differences.

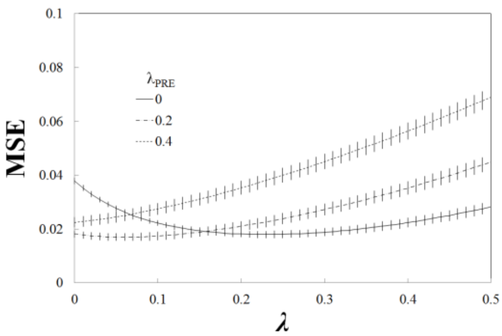

Fig. 10 Mean square error as a function of the parameter λ for λ_{PRE}= 0, 0.2, 0.4, if we set to σ= 0.2.

through the procedure of optical measurements. Also, we have found that the prior information enhancing smooth structures in original wave-fronts was useful for reducing noises which corrupted the original wave-fronts, if we set parameters appropriately. Then, we have found that the MAP estimation using the CG method was also useful for phase unwrapping even for the case that aliasing occurred at several sampling

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