International Journal of
Science and Engineering Investigations vol. 2, issue 12, January 2013
ISSN: 2251-8843

Robust Fuzzy C-Means Clustering with Spatial Information for Segmentation of Brain Magnetic Resonance Images

Saeed Fazli[1], Saeed Fathi Ghiri[2]
[1] Research Institute of Physiology and Biotechnology University of Zanjan, Zanjan, Iran
[2] Dept. of Electrical Engineering, Faculty of Engineering, University of Zanjan, Zanjan, Iran
([1]fazli@znu.ac.ir, [2]sa.fathi@znu.ac.ir)

Abstract- Generalized fuzzy c-means clustering algorithm with improved fuzzy partitions (GIFP_FCM) is a fuzzy clustering algorithm. GIFP_FCM has not a satisfactory performance in image segmentation when the image is contaminated by noise because of not taking into account any spatial information contained in the pixels. In order to solve this problem, a novel robust fuzzy c-means algorithm with spatial information (RFCM_SI) is proposed in this paper. In the proposed method, a novel nonlocal adaptive spatial constraint term is used to modify the objective function of GIFP_FCM. The characteristic of this technique is that the adaptive spatial parameter for each pixel is designed to make the non-local spatial information of each pixel playing a positive role in guiding the noisy image segmentation. Segmentation experiments on synthetic and real images, especially brain magnetic resonance (MR) images, are performed to assess the performance of an RFCM_SI in comparison with GIFP_FCM and fuzzy c-means clustering algorithms with local spatial constraint. Experimental results show that the proposed method is robust to noise in the image and more effective than the comparative algorithms.

Keywords- *Image segmentation; Fuzzy clustering algorithm; Non local spatial constraint; Adaptive spatial parameter; Brain magnetic resonance (MR) image*

I. INTRODUCTION

Magnetic Resonance Imaging (MRI) has provided an effective diagnostic tool that facilitates analysis of pathology for diagnosis, surgery and treatment planning [1-4]. Recently, brain MR images have been widely utilized for detection of any brain disorder.

Brain MRI segmentation to its consisting regions allows radiologists to visualize the individual brain anatomical structures in two or three dimensions. Utilizing the segmentation results physicians are able to diagnose so many dangerous diseases such as multiple sclerosis (MS), schizophrenia, epilepsy, Parkinson's disease, Alzheimer's disease and etc. This makes segmentation a crucial task. Manual segmentation of MR images by radiologists is a challenging, time consuming and costly task. During manual segmentation many errors can occur due to poor hand-eye coordination, low tissue contrast, unclear tissue boundaries caused by partial volumes and operator interpretation.

In the past few decades, numerous segmentation algorithms have been developed [5-9]. Fuzzy C-Means (FCM) is one of fuzzy clustering algorithms which has been widely used for image segmentation. FCM proposes a fuzzy belongingness of each image pixel to each image regions, this causes retaining more information from the image than hard clustering. The original FCM [10] do not consider any spatial information of the image, so it is very sensitive to noise and it has low performance on images which contaminated by noise. To overcome this drawback some modified versions of FCM have proposed that use spatial information in the image. Ahmed et al. [11] proposed Fuzzy C-Means with Spatial constraints (FCM_S), the objective function of FCM_S consists of a spatial neighborhood term. Chen and Zhang [12] proposed two variants of FCM_S: FCM_S1 and FCM_S2. These two algorithms utilize the mean and median gray values of the adjacent pixels of each pixel respectively. This values substitute the neighborhood term of the objective function of FCM_S. Then Szilagyi et al. [13] presented an enhanced fuzzy c-means clustering algorithm (EnFCM). In the EnFCM a linearly-weighted sum image is first formed from both the original image and neighborhood average gray value of each pixel, and then clustering is performed on the gray level histogram of the linearly-weighted sum image instead of the pixels in the summed image, which results in the acceleration of the image segmentation. Subsequently, Cai et al. [14] proposed a fast generalized fuzzy c-means clustering algorithm (FGFCM). The FGFCM performs clustering on the gray level histogram of a novel non-linearly-weighted sum image. This non-linearly-weighted sum image is constructed from both the original image and the spatial coordinates and the gray values within the neighborhood window around each pixel.

Zhu et al. proposed generalized fuzzy c-means clustering algorithm with improved fuzzy partitions (GIFP_FCM) [15] as a novel fuzzy clustering algorithm. In GIFP_FCM, they introduced a membership constraint function and constructed an objective function according to that and furthermore, GIFP-FCM clustering is derived.

The rest of this paper is organized as follows. Section 2 reviews some fuzzy clustering algorithms. Robust Fuzzy C-

Means algorithm with Spatial Information (RFCM_SI) for image segmentation is proposed in Section 3. In Section 4, the performance of RFCM_SI is discussed and the segmentation results of synthetic and real images, especially brain MR images, are presented. Finally, some discussions and concluding remarks are given in Section 5.

II. Fuzzy clustering algorithms

Fuzzy c-means clustering algorithm (FCM) is one of the most popular fuzzy clustering algorithms. $X=\{x_1,x_2,....,x_N\}$ be an image with N pixels, where x_j represents the gray value of the *j*th pixel. The original FCM [10] aims at partitioning X into c clusters by minimizing the following objective function

$$J_m=\sum_{i=1}^{c}\sum_{j=1}^{N}u_{ij}^{m}\left\|x_j-v_i\right\|^2 \quad (1)$$

where,

$$\sum_{i=1}^{c}u_{ij}=1,\ \ u_{ij}\in[0,1],\ \ \ 0\le\sum_{j=1}^{n}u_{ij}\le N \quad (2)$$

where v_i denotes the center of *i*th cluster , u_{ij} represents the belongingness amount of *j*th pixel to *i*th cluster and the parameter *m* is a weighting exponent which determines the fuzziness amount. $\|\ \|$ denotes the Euclidian norm.

Due to not considering spatial information the FCM is very sensitive to noise. In order to solving this problem Ahmed et al. proposed a modified FCM. They changed the objective function as follows:

$$J_m=\sum_{i=1}^{c}\sum_{k=1}^{N}u_{ik}^{m}\left\|x_k-v_i\right\|^2+\frac{\alpha}{N_R}\sum_{i=1}^{c}\sum_{k=1}^{N}u_{ik}^{m}\sum_{x_r\in N_k}\left\|x_k-v_i\right\|^2 \quad (3)$$

where constraints of equation (1) must be satisfied.

Generalized fuzzy c-means clustering algorithm with improved fuzzy partitions (GIFP_FCM) which is proposed by Zhu et al. [15] is a modified version of FCM. Through introducing a novel term for the membership of a pixel to force a more crisp assignment, GIFP_FCM with an appropriate parameter can converge more rapidly than an FCM. They introduced following objective function for the GIFP_FCM :

$$J_m=\sum_{i=1}^{c}\sum_{j=1}^{N}u_{ij}^{m}\left\|x_j-v_i\right\|^2+\sum_{j=1}^{N}a_j\sum_{i=1}^{c}u_{ij}\left(1-u_{ij}^{1-m}\right) \quad (4)$$

where constraints of equation (1) must be satisfied. Minimizing equation (4), the updating equation of membership function u_{ij} and cluster centre v_i achieves as follows:

$$u_{ij}=\frac{1}{\sum_{l=1}^{c}\left(\left(\left\|x_j-v_i\right\|^2-a_j\right)\Big/\left(\left\|x_j-v_l\right\|^2-a_j\right)\right)^{\left(1/(m-1)\right)}} \quad (5)$$

$$v_i=\frac{\sum_{j=1}^{N}u_{ij}^{m}x_j}{\sum_{j=1}^{N}u_{ij}^{m}} \quad (6)$$

where $a_j=\alpha\times\min\left\{\left\|x_j-v_l\right\|^2 \middle| l\in\{1,2,...,c\}\right\}$ controls the u_{ij} to be in $[0,1]$ interval and parameter α controls the convergence speed of this algorithm.

III. Robust Fuzzy C-Means algorithm with Spatial Information (RFCM_SI)

By considering equation (4) we see that the GIFP_FCM doesn't take any kind of spatial information into account and like the original FCM is sensitive to noise. To solve this problem we utilize non local spatial information which is proposed in nonlocal means algorithm for denoising images [16]. Incorporating this non local information and GIFP_FCM we obtained RFCM_SI which has better performance.

A. Non Local spatial Information

The local spatial information derived from the image, such as the mean and median of neighboring pixels within a specified window around each pixel, is often incorporated into fuzzy clustering algorithms [12-14] to make the clustering algorithms robust to noise. These algorithms have good performance on low noise images. However, when the number of noisy pixels in an image is high or the noise level in the image is high the adjacent pixels of a pixel in the image may also contaminated by noise. In this condition, the local spatial information derived from the image cannot play a positive role in making the noisy image segmentation robust.

For each pixel in an image, there are a set of pixels with a similar neighborhood configuration of it which is called Non Local Means [16]. The NL-means not only compares the grey level in a single point but the geometrical configuration in a whole neighborhood. It seems more reasonable to utilize NL-means rather than only local adjacent pixels to obtain spatial information of the image.

We can obtain over desired spatial information for *j*th pixel, $\bar{x}_j$, using the following formula:

$$\bar{x}_j=\sum_{p\in W_j^r}w_{jp}x_p, \quad (7)$$

where W_j^r is a search window centered at *j*th pixel with radius *r*. The weights $0\le w_{ip}\le 1,(p\in W_j^r)$ are adjusted according to the gray level similarity between *i*th and *p*th pixels. The must be assign in such a way that $\sum_{p\in W_j^k}w_{jp}=1$. The similarity between pixels is measured by a weighted Euclidian distance:

$$\left\|x(N_j)-x(N_p)\right\|_{2,\rho}^2=\sum_{q=1}^{(2s+1)^2}\rho^{(q)}\left(x^{(q)}(N_j)-x^{(q)}(N_p)\right)^2 \quad (8)$$

where $x(N_j)$ is a gray level vector of pixels within square window N_j centered at the jth pixel of size $(2s+1)\times(2s+1)$ and $x^{(q)}(N_j)$ is its qth component. $\rho^{(q)}$ is defined by

$$\rho^{(q)} = \sum_{t=\max(d,1)}^{s} \frac{1}{(2t+1)^2} \ , \qquad (9)$$

$$d = \max(|y_q - s - 1|, |z_q - s - 1|)$$

where (y_q, z_q) are the coordinates of the qth component in the defined window. Because we use fixed size window the $\rho^{(q)}$ can be computed in advance as a kernel. The equation (9) illustrates that by increasing distance to the center of window, $\rho^{(q)}$ decreases, so the farther pixels with similar gray levels will get lower weights. The weight w_{jp} is defined as follows

$$w_{jp} = \frac{1}{z_j}\exp(-\left\| x(N_j) - x(N_p) \right\|_{2,\rho}^2 \Big/ h^2) \qquad (10)$$

where z_j is a normalizing constant

$$z_j = \sum_{p\in W_j^r} \exp(-\left\| x(N_j) - x(N_p) \right\|_{2,\rho}^2 \Big/ h^2) \qquad (11)$$

and the parameter h acts as a degree of filtering. It controls the decay of the exponential function and so the decay of the weights as a function of the Euclidean distances.

B. Design of objective function and adaptive spatial parameter in RFCM_SI

In this paper we introduce a novel objective function for fuzzy clustering algorithm utilizing non local spatial information. The objective function of RFCM_SI is presented as follows

$$J_m = \sum_{i=1}^{c}\sum_{j=1}^{N} u_{ij}^m \left\| x_j - v_i \right\|^2 + \sum_{i=1}^{c}\sum_{j=1}^{N} [a_j u_{ij}(1-u_{ij}^{m-1}) + \xi_j u_{ij}^m \left\| \bar{x}_j - v_i \right\|^2] \qquad (12)$$

where $\bar{x}_j$ is non-local spatial information and constraints of equation (2) must be satisfied. The parameter ξ_j is weight that controls the penalty effect for the jth spatial constraint term and is computed as follows

$$\xi_j = \lambda \exp(-\sigma_{x(N_j)}^2 \Big/ \beta X_m) \qquad (13)$$

where $\sigma_{x(N_j)}^2$ and X_m are the standard deviation and mean of pixels gray level within the window W_j^r ,respectively. The parameter β controls the decay of the exponential function and so the decay of the weights which are assigned to each window Equation (13) denotes that the windows which have more similar pixels will get higher weights than others. In the other words ξ_j depends on the homogeneity amount of pixels within the window.

By minimizing Eq.(12) using the Lagrange multiplier method, the update equations of membership function uij and the cluster center vi are given in Eqs. (14) and (15),

$$u_{ij} = \frac{1}{\sum_{l=1}^{c}\left(\dfrac{\left\| x_j - v_i \right\|^2 - a_j + \xi_j \left\| \bar{x}_j - v_i \right\|^2}{\left\| x_j - v_l \right\|^2 - a_j + \xi_j \left\| \bar{x}_j - v_l \right\|^2} \right)^{1/(m-1)}} \qquad (14)$$

$$v_i = \frac{\sum_{j=1}^{N} u_{ij}^m (x_j + \xi_j \bar{x}_j)}{\sum_{j=1}^{N} (1 + \xi_j u_{ij}^m)} \qquad (15)$$

In order to force u_{ij} to be in the [0,1] interval parameter a_j must be defined as follows

$$a_j = \alpha \times \min\{\left\| x_j - v_t \right\|^2 + \xi_j \left\| \bar{x}_j - v_t \right\|^2 | t \in \{1,2,...,c\}\} \qquad (16)$$

where $0 < \alpha < 1$.

TABLE 1 COMPARISON OF THESE FOUR METHODS ON A SYNTHETIC IMAGE CORRUPTED BY GAUSSIAN NOISE.

Methods	Gaussian noise	CA	V_{pc}	V_{pe}
FCM_S1	(0,0.01)	0.9206	0.5142	0.9258
	(0,0.02)	0.8301	0.4615	1.0139
	(0,0.03)	0.7716	0.4369	1.0556
FCM_S2	(0,0.01)	0.9241	0.5157	0.9232
	(0,0.02)	0.8302	0.4612	1.0142
	(0,0.03)	0.7651	0.4385	1.0530
GIFP_FCM	(0,0.01)	0.8610	0.9014	0.2319
	(0,0.02)	0.7459	0.8910	0.2528
	(0,0.03)	0.6748	0.8887	0.2574
RFCM_SI	(0,0.01)	0.9992	0.9995	0.0021
	(0,0.02)	0.9957	0.9976	0.0078
	(0,0.03)	0.9408	0.9908	0.0226

C. Description of the proposed RFCM_SI

The details of robust fuzzy c-means algorithm with spatial information (RFCM_SI) for image segmentation are described as follows.

Step 1: Set the parameters r, s, λ, α, h.
Step 2: Fix the number of clusters c.
Step 3: Set the threshold ε and the maximum iteration number T.
Step 4: Initialize the cluster centers $V = \{v_1, v_2, ..., v_c\}$ and set the iterative index p=1.

Step 5: obtain the non-local spatial information of each pixel using Eqs. (7), (9), (11) and compute ξ_j for each pixel using Eq. (13).
Step 6: Compute the membership functions u_{ij} using Eq.(14).
Step 7: Compute the cluster centers v_i using Eq.(15).

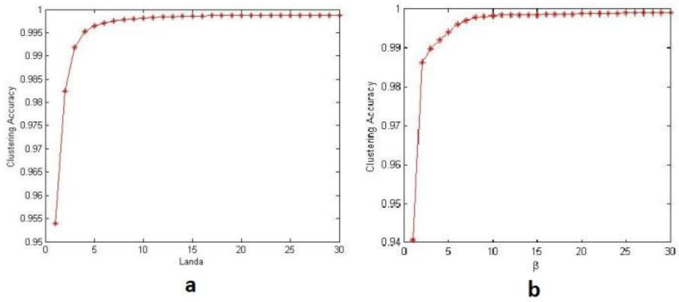

Figure 1: CA versus β and λ; (a) CA versus λ on the synthetic image with Gaussian noise (0,0.01): (b) CA versus β on the synthetic image with Gaussian noise (0,0.01)

Step 8: if $\left|V^{(p+1)} - V^{(p)}\right|^2 < \varepsilon$ or the number of iterations $p < T$, then output the clustering result, otherwise *p=p+1*, goto Step 6.

IV. EXPERIMENTAL RESULTS

In this section, we perform experiments on a synthetic and two real brain MR images, to demonstrate the performance of our proposed algorithm, RFCM_SI. We consider three popular fuzzy clustering algorithms GIFP_FCM [15], FCM_S1 [12] and FCM_S2 [12] as comparative methods. A 3×3 window around each pixel in the image is taken as the local information of the pixel in order to calculate the mean and median for FCM_S1 and FCM_S2, respectively. We assume the fuzziness index *m* is equal to 2 and the maximum iteration number T and the threshold ε are set to 2500 and 10^{-5}, respectively. As discussed in [15] in detail selection of the parameter $\alpha > 0.9$ satisfies clustering performance and fast convergence speed. So it is set to 0.99 for GIFP_FCM and RFCM_SI.

As presented in Eq. (13), the adaptive spatial parameter ξ_j of the *j*th pixel is related to λ, β and its neighborhood pixels. It can be get from Eq. (13) that as λ decrease, the spatial constrains effect decrease too. Too small λ may make the spatial constraints of some pixels unable to have positive role in the pixel clustering, and too large λ may lead to the pixel clustering heavily relying on the spatial constraint. For determining a reasonable choice to λ we set $\beta = 10$ and test λ on the interval [1 30]. Fig.1a illustrates the effect of λ on the Clustering Accuracy (CA) [17]. So we set $\lambda = 20$ in the rest of this paper.

Fig.1b illustrates the dependence of CA to β. For determining a reasonable choice to β we set $\lambda = 20$ and test β on the interval [1 30]. So we set $\beta = 30$ in the rest of this paper.

To compute CA for a clustering result, we need to build a permutation mapping function *map(·)* that maps each cluster index to a true class label. The clustering accuracy based on *map(·)* can then be computed as follows

$$CA = \frac{\sum_{j=1}^{N} \delta(y_j, map(c_j))}{N} \quad (17)$$

where y_j and c_j are the true class label and the obtained cluster index of *j*th pixel respectively, $\delta(x, y)$ is a function that equals 1 if x = y and equals 0 otherwise.

We use two other performance measures to quantitatively assess these four methods, the partition coefficient *Vpc* [18] and the partition entropy *Vpe* [19] are adopted to evaluate the segmentation results. The partition coefficient *Vpc* and the partition entropy *Vpe* are two cluster validity functions, which are defined as follows:

$$V_{pc} = \sum_{i=1}^{c} \sum_{j=1}^{N} u_{ij}^2 / N \quad (18)$$

$$V_{pe} = -\sum_{i=1}^{c} \sum_{j=1}^{N} (u_{ij} \ln u_{ij}) / N \quad (19)$$

The idea of these two validity functions is that the partition with less fuzziness means better performance. So the best clustering is achieved when *Vpc* is going to be one (maximum) and *Vpe* is going to be zero (minimum).

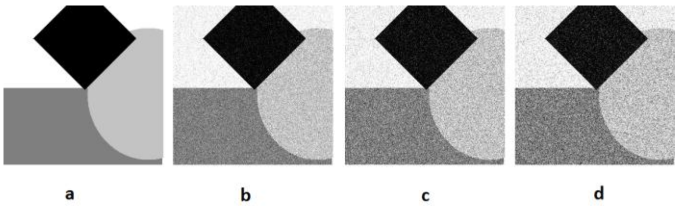

Figure 2: (a): the original synthetic image; (b),(c) noisy image with Gaussian noise (0, 0.01); (c) noisy image with Gaussian noise (0, 0.02); (d) noisy image with Gaussian noise (0, 0.03)

In this paper we test these four methods on a 281×281 pixel synthetic image. This image includes four clusters with gray levels 0, 127, 195 and 255. In order to evaluate these methods in presence of noise we add Gaussian noise to our test images. We utilize MATLAB function imnoise for adding noise to the test images. Fig.2 illustrates the original synthetic image and the noisy ones which has contaminated by Gaussian noise of mean 0 and variance 0.01, 0.02 and 0.03 respectively.

The segmentation results on the synthetic image which is corrupted by Gaussian noise are shown in Fig. 3. It can be found from Fig.3 that our proposed RFCM_SI has the best performance and it is robust enough encountering noisy images. The quantitative comparison among these methods is presented in table. 1. The results show that RFCM_SI has better performance.

TABLE 2: COMPARISON OF THESE FOUR METHODS ON TWO REAL IMAGES CORRUPTED BY NOISE.

	Noise	Methods	FCM_s1	FCM_s2	GIFM_FCM	RFCM_SI
MRI 1	Gaussian (0,0.2)	V_{pc}	0.4489	0.4476	0.8997	**0.9901**
		V_{pe}	1.0310	1.0326	0.2349	**0.0244**
MRI 2	Rician	V_{pc}	0.3971	0.3976	0.8794	**0.9932**
		V_{pe}	1.1104	1.1091	0.2745	**0.0172**

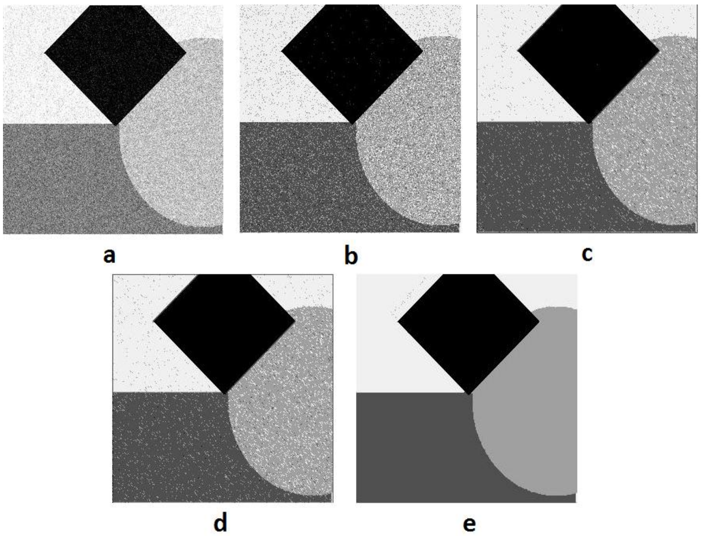

Figure 3: segmentation of the synthetic image corrupted by Gaussian noise (0,0.01) (a) noisy image; (b) FCM_S1; (c) FCM_S2; (d) GIFP_FCM; (e) RFCM_SI.

We test our proposed algorithm on two brain MR images. These images are T1 and T2 weighted images which are available at http://www.med.harvard.edu/AANLIB. It is known that MR images are usually contaminated with Rician noise. Therefore we add Rician noise to one of this MR images and Gaussian noise to the other one in order to show performance and robustness of our algorithm. Fig 4 illustrates the segmentation results on a T2 weighted brain MR image which has contaminated by Rician noise of level 20. Fig.5 shows the segmentation results on a T1 weighted brain Mr image which has contaminated by Gaussian noze of mean zero and variance 0.02.. The results show that RFCM_SI has the best performance encountering both kind of noises on synthetic a real images.

V. CONCLUSION

In order to overcome the sensitivity of GIFP_FCM to noise in brain MR images, a novel robust fuzzy c-means with spatial information (RFCM_SI) is proposed in this paper. This method utilizes the non-local spatial information of each pixel in the image to guide the noisy image segmentation. The experimental results show that an RFCM_SI can obtain satisfying segmentation performance on brain MR images which are contaminated by noise. In this paper, some parameters of an RFCM_SI are preliminarily discussed in the experiment section. Thus, how to theoretically choose these parameters deserves to be studied.

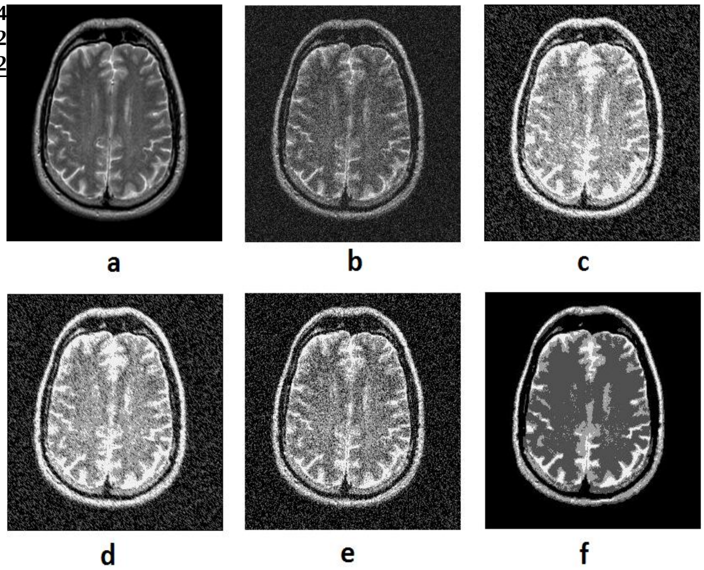

Figure 4: segmentation of the synthetic image corrupted by Rician noise of level 20 (a)original image; (b) noisy image; (c) FCM_S1; (d) FCM_S2; (e) GIFP_FCM; (f) RFCM_SI.

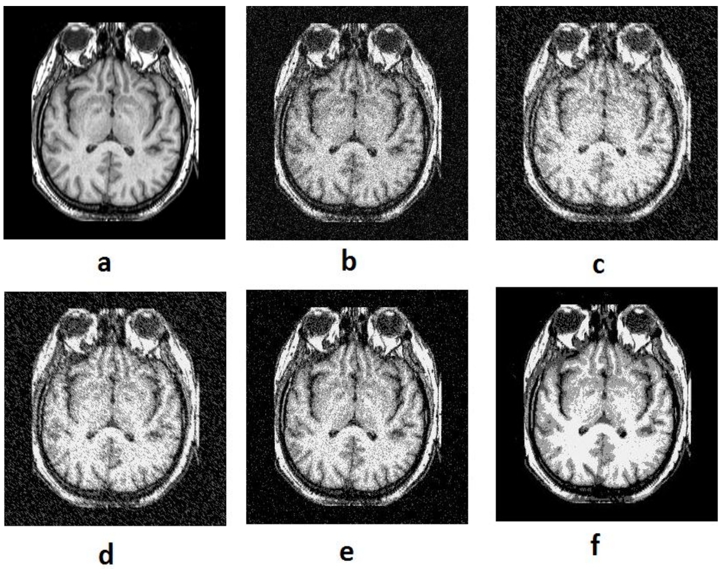

Figure 5 : segmentation of the synthetic image corrupted by Gaussian noise (0,0.2) (a)original image; (b) noisy image; (c) FCM_S1; (d) FCM_S2; (e) GIFP_FCM; (f) RFCM_SI

REFERENCES

[1] K. Jain, Fundamentals of Digital Image Processing, Prentice-Hall,Upper Saddle River, NJ, 1989.

[2] R.C. Gonzalez, R.E. Woods, Digital Image Processing, Pearson Education India, Fifth Indian Reprint, 2000.

[3] N.R. Pal, S.K. Pal, A review on image segmentation techniques, Pattern Recognition 26 (1993) 1277–1294.

[4] Juin-Der, Hong-Ren Su, Philip E. Cheng, Michelle Liou, John A.D.Aston, Arthur C. Tsai, Cheng-Yu Chen, MR image segmentation using a power transformation approach, IEEE Transactions on Medical Imaging 28 (6) (2009) 894–905.

[5] G.S.Linda, C.S.George, Computer Vision, Prentice-Hall, New Jersey, 2001.

[6] H.Zhang, J.E.Fritts, S.A.Goldman, Imagesegmentationevaluation: asurvey of unsupervised methods, Comput. Vision Image Understanding 110 (2008) 260–280.

[7] P.F.Felzenszwalb, D.P.Huttenlocher, Efficient graph-based image segmentation, Int. J.Comput.Vision 59 (2) (2004) 167–181.

[8] Y.Zhou, J.Starkey, L.Mansinha, Segmentation of petrographic images by integrating edge detection and region growing, Comput.Geosci.30 (2004) 817–831.

[9] S.Osher, N.Paragios, Geometric Level Set Methods in Imaging, Vision, and Graphics, Springer-Verlag, NewYork, 2003.

[10] J.C. Bezdek, Pattern Recognition with Fuzzy Objective Function Algorithms, Plenum, New York, 1981.

[11] M.N. Ahmed, S.M. Yamany, N. Mohamed, A.A. Farag, T. Moriarty, A modified fuzzy c-means algorithm for bias field estimation and segmentation of MRI data, IEEE Trans. Med. Imaging 21 (2002) 193–199.

[12] S.C. Chen, D.Q. Zhang, Robust image segmentation using FCM with spatial constraints based on new kernel-induced distance measure, IEEE Trans. Syst. Man, Cybern. 34 (2004) 1907–1916.

[13] L. Szilagyi, Z. Benyo, S. Szilagyii, H.S. Adam, MR brain image segmentation using an enhanced fuzzy C-means algorithm, In: Proceedings of the 25th Annual International Conference of the IEEE EMBS, Cancun, Mexico, 2003, pp. 17–21.

[14] W. Cai, S. Chen, D. Zhang, Fast and robust fuzzy c-means clustering algorithms incorporating local information for image segmentation, Pattern Recognit. 40 (2007) 825–838.

[15] L.Zhu, F.L.Chung, S.T.Wang, Generalized fuzzy c-means clustering algorithm with improved fuzzy partitions, IEEE Trans. Syst. ,Man, Cybern. B,Cybern. 39 (3)(2009)578–591.

[16] Buades, B. Coll, J.M. Morel, A non-local algorithm for image denoising. In : Proc. IEEE Int. Conf. Comput. Vision Pattern Recognition. (2005) 60–65.

[17] M. Wu, B. Scholkopf, A local learning approach for clustering. in: Advances in Neural Information Processing Systems 19. (2007) 1529–1536.

[18] J.C. Bezdek, Cluster validity with fuzzy sets, Cybern. Syst. 3 (3) (1973) 58–73.

[19] J.C. Bezdek, Mathematical models for systematic and taxonomy. In: Proc. of eighth international conference on numerical taxonomy. (1975) 143–166.