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Calculating Position of Feed Antenna and Dimensions of Subreflector in the Cassegrain Antenna Using the Bisection Numerical Method

Ali-Reza Sharifi
Faculty of Engineering, University of Zanjan, Zanjan, Iran
(arsharifi@znu.ac.ir)

Abstract- In this paper Cassegrain antenna and its geometric structure is reviewed and a relation for calculating dimensions of its main and subreflector to minimize the blockage is investigated. This relation which is a complicated equation with respect to several variables is then simplified in the design example and expressed versus only one variable. The simplified equation is then solved numerically using the method of bisection with rapid convergence and high precision. The design example clarifies the method more clearly.

Keywords- *Cassegrain antennas, Bisection numerical method, root finding methods, optimization, dual reflector antennas*

I. INTRODUCTION

Dual reflector antennas are wideband and have pencil beam and high gain, so they are very common in space communication and millimeter wave radio telescopes [1]. Fig. 1 shows a prototypical Cassegrain telescope which is located in West Virginia[1]. These antennas also have applications in the structure of instruments which produce plane wave as millimeter wave compact antenna test range (CATR) devices [2]. Considering the large dimensions of these antennas ;with main reflector diameter usually in the range of 10 meters and more, their analysis is carry out using the physical optics (PO) method, and other analytical methods such as the method of finite elements and the method of moments would be very time consuming and require very large computation volume. Design equations of this structure are mainly based on the mathematical equations of parabolic and hyperbolic surfaces.

In reflector antennas usually the field strength in the apex of reflector is not the same as its value at the edge of reflector, and this is called tapering the filed strength. Tapering has direct impact on the beam width of the main lobe and side lobe levels.

Fig. 2 depicts the calculated power pattern of a prime reflector antenna in logarithmic scale with respect to $u=(\pi d/\lambda)\sin(\theta)$ for various tapering values. θ is the space angle with respect to the axes of parabolic reflector[1]. Narrowest beam width corresponds to the case of uniform field distribution and have largest side lobe level value of about (-17dB), while the broadest beam width is related to the case which the amount of field at the edge of reflector is tapered to zero with respect to the field in the apex of reflector. Anyway this case has the lowest side lobe level in the order of -24 dB, and may have advantages in some applications. In plotting this figure the field distribution of the aperture plane is considered as a quadratic function with respect to r; the radius of aperture plane; and also is a function of the tapering factor τ , as the following equation:

$$F(r)=\tau+(1+\tau)(1+r^2)=1-(1-\tau)r^2 \qquad (1)$$

Figure 1. schematic of the national radio astronomy observatory (NRAO) telescope, located in West Virginia, USA. The reflectors are made from aluminum

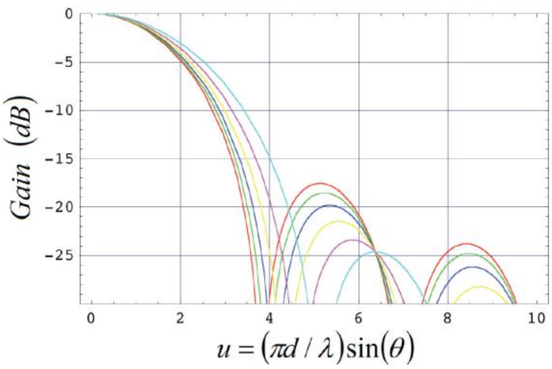

Figure 2. Variation of main lobe beam width and side lobe level of prime reflector parabolic antenna with respect to different tapering factors.

Other than the above effects, tapering will have direct influence on aperture efficiency and received noise level by antenna from unwanted space directions. Reflector antenna gain is calculated from the following,

$$G = \eta_A \frac{4\pi}{\lambda^2} A_p \tag{2}$$

Where A_p is the physical aperture area, λ is the free space wavelength and η_A is the aperture efficiency which is calculated approximately from:

$$\eta_A = \eta_i \eta_s \eta_p \eta_b \tag{3}$$

In (3), η_i is called illumination efficiency of aperture and relates directly to the field tapering and is calculated by:

$$\eta_i = \frac{\left(\int F(r,\phi)\, dA\right)^2}{\int F^2(r,\phi)\, dA} \tag{4}$$

surface integral is taken over the area of aperture plane and $F(r,\varphi)$ is the field distribution on the aperture surface. As is seen from Fig. 2, tapering the field distribution reduces the gain and indicates that η_i reduces by introducing more taper.

In (3), η_s is spillover efficiency and shows the ratio of feed energy that reaches to the reflector surface to total energy radiated by the feed. So by increasing the edge taper value η_s increases too and less energy would be dissipated in unwanted directions. η_s and η_i act in reverse manner by changing the edge taper value but the product $\eta_i \eta_s$ peaks to 0.8 for the edge taper of about -12 dB.

In (3), η_p is the cross polarization efficiency and depends on the unwanted components of field perpendicular to the main field direction. Increasing these components would result in reduction of η_p. Cross polarization reduction is one of the challenges every one encounters while designing reflector antenna systems. Specially in tracking applications, the presence of unwanted perpendicular field components would increase nothing but extra received noise. This factor depends on several parameters like the ratio of focal length to the antenna diameter, field distribution of feed antenna, roughness of the reflector surface and etc. Dual reflector antennas like Cassegrain antenna have less cross polarization component since they have larger effective ratio of focal length to main reflector diameter and have larger η_p.

In (3), η_b is the blockage efficiency and is due to blocking the waves by obstacles placed in front of the main reflector, like the feed antenna or subreflector, etc. For acceptable η_b in reflector antennas, diameter of subreflector or feed antenna placed in front of the main reflector must be less than 0.1 diameter of the main reflector antenna. To obtain optimum η_b in Cassegrain antenna, subreflector diameter may be equal to the shadow made by feed antenna on the reflector surface.

In this paper after introducing the mathematical relations that govern the conical surfaces and Cassegrain antenna, method of bisection in solving nonlinear equations is reviewed in sections 2 and 3 respectively. Design example of this kind of antennas is presented in section 4 to illustrate the problem more clearly.

II. CONICAL SECTIONS

Parabola, hyperbola and ellipse are created when a plane crosses a cone in different directions. Generally speaking, a conical surface or curve is the locus of all points that their distance from a fixed point called focal point be e times their distance from a fix line, called directrix. Coefficient e is called eccentricity. In ellipse e is less than 1, while in parabola it equals 1 and in hyperbola is greater than 1. ellipse and hyperbola have two focal point while parabola have one. Fig. 3 depicts a conical curve. In this figure, conical point is presented by F. the distance of F to directrix line is assumed equal to c, while the distance of curve apex to directrix line is taken equal to a, as indicated in the figure. Parabolic or hyperbolic equation in polar coordinate system using the above definition is exploited here.

Assume that the origin point be located on F and the axes of the curve be placed on x axes. Considering Fig. 3, we have,

$$\begin{aligned} &XF = eXZ \quad \Rightarrow r = e\left(c - r\cos(\theta)\right) \\ &\Rightarrow r\left(1 + e\cos(\theta)\right) = ec \end{aligned} \tag{5}$$

so,

$$r = \frac{ec}{\left(1 + e\cos(\theta)\right)} \tag{6}$$

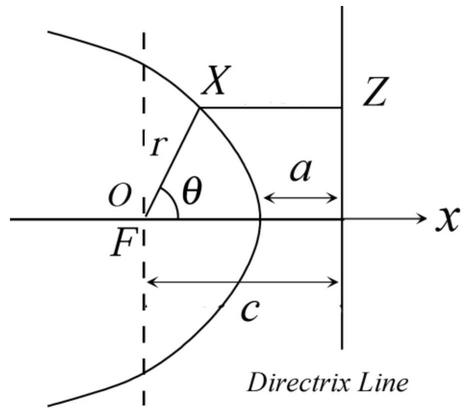

Figure 3. A hyperbolic or parabolic conical section which its focal point, F, is located on the Origin, O, and its axes is placed on x-axes, directrix line is parallel to the y-axes. The conical section is a parabola if e=1 and a hyperbola if e>1.

In a parabola, e=1, c=2a . so the above equation becomes,

$$r = \frac{2a}{\left(1+\cos\left(\theta\right)\right)} \tag{7}$$

while in the hyperbola, (6) turns into,

$$e = \left(c-a\right)/a \Rightarrow c = a\left(e+1\right) \Rightarrow r = \frac{ae\left(e+1\right)}{1+e\cos\left(\theta\right)} \tag{8}$$

By rotating parabola or hyperbola along its axes by 360 degree, parabolic or hyperbolic surfaces create, which have wide applications in designing Cassegrain antennas.

III. GEOMETRIC STRUCTURE OF CASSEGRAIN ANTENNAS

In Fig. 4, geometric structure of Cassegrain antenna is indicated. This structure is created from a parabolic main reflector with diameter D_P and focal length F_P, a hyperbolic subreflector with diameter D_S and focal length F_S and a feed antenna. Phase center of the feed antenna is presented by the left cross sign in the figure and is located on one of the focal points of the hyperbolic surface. another focal point of the subreflector is illustrated by the right cross sign in the figure and is located on the focal point of the parabolic main reflector. The angle of edge rays hit from feed antenna to the subreflector is indicated by θ_0, and that's of main reflector is stated by ψ_0. Feed antenna is assumed here as conical corrugated type, which has less cross polarization field components with respect to ordinary circular horn antenna. It has almost the same E and H pattern. In the figure, S_F is a parameter which presents the amount of displacing the phase center of the feed with respect to the feed aperture. It depends on the slant radius of horn and wavelength, etc. waves leaving the feed antenna, hit the subreflector as plane waves.

From Fig. 4 and equ. (7), ψ_0 and θ_0 can be obtained versus F_p / D_p and F_e / D_p respectively. In the main reflector, distance between focal point and the edge of reflector is obtained from (7) as,

$$r_0 = \frac{2F_p}{\left(1+\cos\left(\psi_0\right)\right)} = \frac{F_p}{\cos^2\left(\frac{\psi_0}{2}\right)} \tag{9}$$

on the other hand,

$$r_0 \sin\left(\psi_0\right) = \frac{D_p}{2} \tag{10}$$

from (9) and (10),

$$\tan\left(\frac{\psi_0}{2}\right) = \frac{D_p}{4F_p} \tag{11}$$

similarly,

$$\tan\left(\frac{\theta_0}{2}\right) = \frac{D_p}{4F_e} \tag{12}$$

The ratio of effective focal length F_e to F_p is called antenna magnification, M. relation between M and eccentricity, e, is,

$$e = \frac{M+1}{M-1} \tag{13}$$

eccentricity also may be obtained versus θ_0 and ψ_0 as,

$$e = \frac{\sin\frac{1}{2}\left(\psi_0+\theta_0\right)}{\sin\frac{1}{2}\left(\psi_0-\theta_0\right)} \tag{14}$$

In designing antenna, the main reflector diameter usually is calculated from communication link budget and mechanical considerations. The ratio of F_p / D_p is usually chosen between 0.25 and 0.5 and M is selected larger than 1 to reduce cross polarization field components. Knowing M, the F_e / D_p ration will be known too. From (11) and (12), ψ_0 and θ_0 is gotten. So the parameters F_p, D_p, ψ_0 and θ_0 in Fig. 4 are all known.

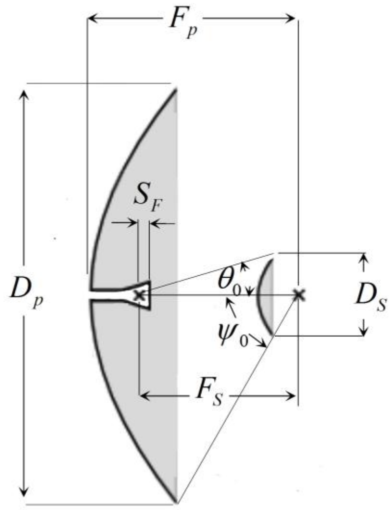

Figure 4. Geometric structure of Cassegrain antenna system.

Now a proper feed antenna must be chosen with less cross polarization field components and with ability to illuminate the edge of reflector with filed taper equal to about -12 dB with respect to reflector center. Considering Fig. 4, it is clear that for fixed ψ_0 and θ_0, whatever the feed be placed farther than subreflector, the subreflector diameter D_S must be larger to be able to cover the beam limits, and that will lead to greater blockage against the main reflector and reduces η_b. On the other side if the feed antenna be located very close to the subreflector, D_S, can be made smaller, but the shadow of the antenna feed created on the main reflector would be large enough to increase the blockage effect again, so in optimum case, the feed position must be determined in such a way that subreflector diameter and the shadow of feed antenna against the main reflector become equal. To do this, we need some extra information about feed antenna, like feed antenna aperture diameter and the phase center position of radiated fields.

Physical shape of a typical feed antenna is shown in Fig. 5. In Cassegrain antennas, the feed may be located in a protection cavity against humidity and rain effects, so its effective diameter may be larger than antenna aperture diameter. In Figs. 5, 7, the sum of feed aperture diameter plus protective cavity thickness is named as the feed diameter, AF.

Corrugations in the internal surface of the horn generate hybrid modes which have lower cross polarization components with respect to dominant TE11 mode in ordinary conical horn. Dimensions of the structure depends on the antenna gain, the required edge tapering off the field, design frequency and mechanical considerations and is based on Fig. 6 [4]. In this figure S parameter is defined as:

$$S = \frac{a^2}{2\lambda R} \tag{15}$$

where a is the radius of feed aperture, λ is the free space wavelength and R is the radius of curvature of the horn.

In designing the feed, S parameter may be chosen arbitrarily, knowing the required amount of edge tapering of the beam and θ_0, from Fig.6, the amount of $\frac{2\pi a}{\lambda}\sin(\theta_0)$ will be known too, and so the feed aperture radius, "a", will be obtained.

Figure 5. Physical structure of a prototypical conical corrugate horn antenna

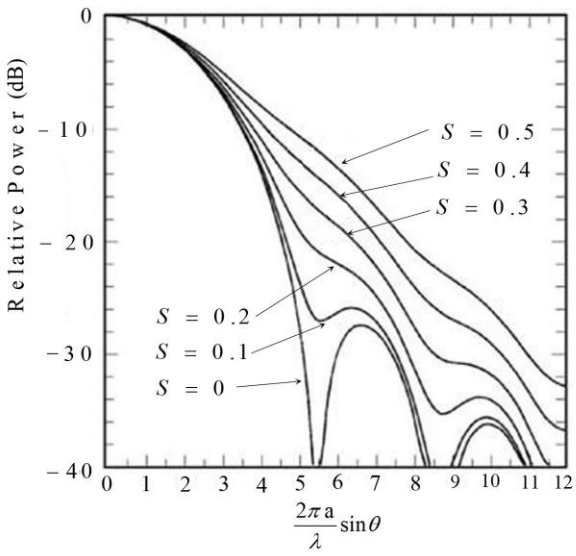

Figure 6. radiated power of the corrugated horn antenna with respect to radiation angle and for various S parameters.

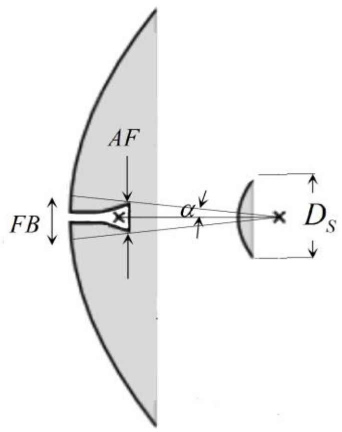

Figure 7. Geometric description of the blockage effect made by feed shadow on the main reflector.

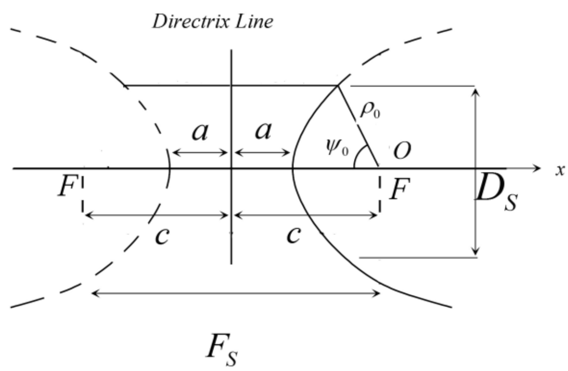

Figure 8. Geometric parameters of hyperbolic section.

Knowing S parameter and feed aperture radius, slant radius of the horn, R, is determined from (15) and the ratio of phase center displacement of the feed, S_F to R is determined from [4]. By obtaining AF and S_F, using the theory of physical optics, a relation can be obtained to equalize the subreflector diameter with the feed shadow on the main reflector.

In Fig. 8, hyperbolic conical section is depicted which consists of two curves and two focal points. Focal point distance is $F_S = 2c$, and the distance between apex of curves is 2a. The left hyperbolic curve which is shown by dashed line in this figure does not exist physically in Fig. 4. The eccentricity coefficient, e, in Fig.8 is obtained as:

$$e = \frac{c-a}{a} \tag{16}$$

To determine the geometric shape of hyperbolic, two parameters, a and c must be known.

To obtain a relation between subreflector diameter, D_S, and ψ_0 angle, we have from (8):

$$\rho_0 = \frac{ae(e+1)}{1+e\cos(\psi_0)} \tag{17}$$

$$\frac{D_S}{2} = \rho_0 \sin(\psi_0) \tag{18}$$

So,

$$D_S = \frac{2ae(e+1)}{1+e\cos(\psi_0)}\sin(\psi_0) \tag{19}$$

To determine diameter of the shadow of the feed on the main reflector, FB, one may use (7) together with Fig.7. α angle is calculated as:

$$\tan(\alpha) = \left(\frac{\frac{AF}{2}}{F_S - S_F}\right) \tag{20}$$

using (7):

$$FB/2 = \frac{2F_p}{(1+\cos(\alpha))}\sin(\alpha) \tag{21}$$

so,

$$FB = \frac{4F_p}{(1+\cos(\alpha))}\sin(\alpha) = 4F_p\tan\left(\frac{\alpha}{2}\right) \tag{22}$$

by equating (22) and (19),

$$\frac{2ae(e+1)}{1+e\cos(\psi_0)}\sin(\psi_0) = 4F_p\tan\left(\frac{a\tan\left(\frac{\frac{AF}{2}}{F_S - S_F}\right)}{2}\right) \tag{23}$$

in (23), $F_S = 2c = 2a(1+e)$, so (23) may be written as

$$f(a) = \frac{2ae(e+1)}{1+e\cos(\psi_0)}\sin(\psi_0) - 4F_p\tan\left(\frac{a\tan\left(\frac{\frac{AF}{2}}{2a(1+e)-S_F}\right)}{2}\right) = 0 \tag{24}$$

As explained earlier, in (24), all parameters except for a is known. equation. (24) is a nonlinear equation with respect to “a” and cannot be solved simply. Among different numerical methods for solving this equation, one must use a simple method with low calculation volume and rapid rate of convergence. The method of Bisection is used here for this purpose. The Bisection method is one of the bracketing methods in which we seek the root of function in a special interval. The algorithm is as follows [5]. At first by sketching the function in a special interval we seek for any point that function crosses the horizontal axes. Then a suitable interval is chosen [j,k], where j and k are points at the left and right of the approximate root respectively and f(j)f(k)<0. The distance between j and k then divides into two sections with the middle section called c. the amount of function in this point, f(c), is then calculated. c will be closer than j and k to the root point. Now if f(c) satisfies the required precision for the problem under hand, we stop at this point, otherwise we choose the new interval [c, k] if f(c)f(k)<0, or [j,c] if f(c)f(k)<0 to seek for the root position again. This process is repeated until the satisfied precision is obtained or the number of iterations exceed than maximum required. In continue we explain all the above mentioned by a numerical example.

IV. NUMERICAL EXAMPLE IN DESIGNING CASSEGRAIN ANTENNA SYSTEM

Assume that the main reflector diameter, D_P, be 10 m considering communication link budget and mechanical limitations. The center frequency of operation for a X-band radar is taken as 10 GHz. We choose $F_p / D_p = 0.5$, so the focal distance of parabolic surface would be 5 (m). By selecting antenna magnification, M, equal to 4:

$$F_e / F_p = 4 \Rightarrow F_e = 20 \ (m)$$

from (13):

$$e = \frac{4+1}{4-1} = \frac{5}{3}$$

from (11), (12), $\psi_0 = 53.1^{\circ}$ and $\theta_0 = 14.2^{\circ}$. By taking S=0.2, for conical corrugated horn feed, and assuming -12 dB edge taper for illumination of reflectors, observing Fig. 6,

$$\frac{2\pi a}{\lambda} \sin \theta_0 = 3.5 \ ,$$

$$\lambda = c / f = 3 \times 10^8 / 10 \times 10^9 = 0.03 \ (m)$$

feed aperture diameter then is:

$$a = 3.5 \times 0.03 / (2\pi \times \sin(14.2^{\circ})) = 6.8 \ (cm)$$

the thickness of shield surrounding the feed is taken equal to 2 cm so,

$$AF = 2(a+2) = 17.6 \ (cm)$$

from (15), R is obtained. As mentioned earlier, R is used in relations for determination of the phase center displacement of the feed, S_F. We have:

$$R = \frac{a^2}{2\lambda S} = \frac{(6.8)^2 \times 10^{-4}}{2 \times 0.03 \times 0.2} = 38 \ (cm)$$

from [4],

$$S_F = R \times 0.124 = 4.7 \ (cm)$$

Now all parameters in (24) are specified except for “a”. To solve this equation by the method of Bisection, $y = f(a)$ is plotted by [6], in the interval [0,5] in Fig. 9. The interval is chosen according the physical structure of the problem. Since we assume that feed antenna is placed in the space between the main reflector and subreflector, considering the origin point located at the focal point of the main reflector, referring to Fig. 8 the amount of 2a would be greater than zero and less than 5 (m) in this problem. To investigate the possibility of existing zero of $y = f(a)$, beyond this distance, we may plot this function in [0, 5]. As seen from Fig. 9, the root is placed in [0.1, 0.4] interval, where f(0.1)f(0.4)<0. The algorithm for root finding by the method of Bisection is implemented in MATLAB software [6]. While running the program, it is assumed that the precision be better than 0.01 or the iteration numbers does not exceed 20 times. After 7 iteration the approximate root, a=0.3086, is obtained. Steps of this process are summarized in Table 1.

After determination of “a”, other physical parameters of the subreflector like D_S, F_S will be obtained . From (19),

$$D_s = \frac{2 \times 0.31 \times (5/3)((5/3)+1)}{1 + (5/3)\cos(53.1^{\circ})} \sin(53.1^{\circ}) = 1.1 \ (m)$$

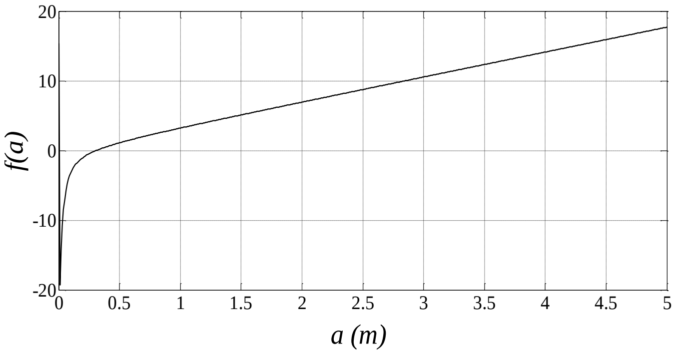

Figure 9. Variation of (24), in the design example versus variable “a” which is half the distance between hyperbola apexes, shown in Fig. 8.

TABLE I. ITERATIONS OF THE IMPLEMENTED ALGORITHM IN SOLVING (24) FOR THE DESIGN EXAMPLE.

iteration	$\lvert f(c) \rvert$	c
1	0.4737	0.25
2	0.1138	0.325
3	0.1585	0.2875
4	0.0180	0.3063
5	0.0489	0.3156
6	0.0157	0.3109
7	0.0011	0.3086

and since $F_s = 2c = 2a(1+e)$, we have:

$$F_s = 2 \times 0.31 \times (1 + (5/3)) = 1.65 \ (m))$$

so the phase center of feed antenna must be 3.35 (m) apart from the main reflector apex. The steps of designing antenna dimensions in this example, is now over.

V. CONCLUSION

In this paper Cassegrain antenna system specifications are reviewed. Using the physical optic method and aiming mathematical relations of parabolic and hyperbolic surfaces, we get a nonlinear and complex function with respect to antenna parameters. This equation is then approximately solved using the method of Bisection. A design example for radar applications in X-band is presented which explains the method clearly.

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