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Numerical Analysis of the Heat Conduction Problem Using the Energy of the Solutions

Alcineide Pessoa[1], Marta Barreiros[2], Gean Sousa[3]
[1,2,3]Federal University of Maranhão
[3]Professor in Federal University of Maranhão
([1]alcineidedutra@hotmail.com, [2]marta-barreiros@hotmail.com, [3]gean_mat@yahoo.com.br)

Abstract- The problem of conducting heat in a homogeneous environment has been studied by several researchers in the fields of engineering and mathematics, mainly by mathematicians who study Numerical Analysis. The classic problem that shapes this phenomenon has undergone several criticisms and proposals for improvement. One of the proposals is the Maxwell-Cattaneo model, which seeks to solve a paradox known as a Fourier paradox. The numerical solution of the Maxwell-Cattaneo law through Finite Differences has also been criticized and proposed solutions. One such proposal is an Unusual Usual Finite Differences scheme. In this paper we propose to make a numerical analysis of the unusual scheme based on the energies of the solutions. We present our results through a proposition and also through graphs. We realize that the graphical analysis collaborates with what was proposed analytically in the proposition.

Keywords- *Heat Conduction, Numerical Analysis, Finite Differences, Maxwell-Cattaneo, Energy*

I. INTRODUCTION

Numerical and Computational Analysis has consolidated as an important area of Mathematics [1,2]. The solution of physical, biological or engineering problems through numerical methods imply the need for a better understanding of the numerical results and, in some cases, the construction of new methods. The construction and analysis of new methodologies is necessary to overcome limitations imposed by the schemes considered as usual. Considering the context of numerical regulation, we can mention the schemes known as non-standards (Non-standard Finite Difference Methods-NSFD) [3-6].

Properties such as solution positivity, solution limitations and monotonicity of solutions when analyzed in the context of NSFDs perform better than classical schemes. This is due to the flexibility that NSFD's have to preserve certain properties that are obeyed by the original mathematical model [5]. This improved performance has been observed in several applications. Among them, we can highlight the application of NSFD to solve problems of population dynamics in biological problems and diffusion-reaction problems with the objective of generating numerical solutions that are positive and limited [7-9].

In this paper we present an analysis of an important result about the use of NSFDs from the work of Mickens and Jordan [5]. In this work, the authors construct a NSFD-like scheme capable of producing with physical consistency heat-wave solutions and of correcting a Physical Paradox imposed by the Fourier Hypothesis in Linear Thermoelasticity. Mickens and Jordan present the model and simulations of the solutions. However, a robust method analysis is not performed.

In order to contribute to a numerical analysis of the scheme presented by [5] it is that in this paper we present a numerical energy proposal associated with the NSFD scheme. From this energy it is possible to make important comparisons and to promote more general observations about the proposed problem. Our results concentrate on presenting an analysis based on the energy behavior of the heat wave propagation system.

We present some mathematical demonstrations and computational simulations for a better visualization of numerical results. Our simulations corroborate with the authors' results about the importance of using NSFDs.

II. METHODS

A mathematical model applied to engineering problems is the expression that mathematically represents the behavior of heat propagation in a homogeneous bar [10,11]. This equation is represented analytically as follows:

$$\frac{d\theta(x,t)}{dt} - k\frac{d^2\theta(x,t)}{dx^2} = 0 \qquad (1)$$

where θ (x, t) describes the absolute temperature at a point x at time t.

According to Mickens and Jordan [5], this model (known as Fourrier's law) implies in the fact that a thermal perturbation at any point in a body will be instantly felt at all other points in the body. That is, this model predicts that thermal signals propagate with infinite velocity, which does not occur in practice. In order to "correct" the problem cited several proposals have been presented, and among these proposals is the law of Maxwell-Cattaneo [12].

The Maxwell-Cattaneo law consists of a hyperbolic system given by:

$$u_t - u_{xx} + \tau_0 u_{tt} = 0, \quad u_t(x,t) \in (0,1)X(0,\infty); \tag{2}$$

$$u(0,t) = u(1,t) = 0\,, \qquad t > 0; \tag{3}$$

$$u(x,0) = \sin[\pi x], \quad u_t(x,0) = 0\,, \qquad x \in (0,1) \tag{4}$$

on what: $u_t = \frac{du(x,t)}{dt}$, $u_{xx} = \frac{d^2u(x,t)}{dx^2}$ e $u_{tt} = \frac{d^2u(x,t)}{dt^2}$.

Using the method of separation of variables we can verify that the analytic solution for this system is given by [13]:

$$u(x,t) = e^{\frac{-t}{2\tau_0}} \sin[\pi x].\begin{cases} \cosh[\omega t] + \dfrac{\sinh[\omega t]}{\sqrt{\Delta}}; \tau_0 < \tau_c \\ 1 + \dfrac{t}{2\tau_0}; \tau_0 = \tau_c \\ \cosh[\omega t] + \dfrac{\sinh[\omega t]}{\sqrt{|\Delta|}}; \tau_0 > \tau_c \end{cases}$$

on what, $\omega = (2\tau_0)^{-1}\sqrt{|\Delta|}$, $\Delta = 1 - 4\pi\tau_0$ e $\tau_c = (2\pi)^{-2}$.

The NSFD scheme proposed by Mickens and Jordan to solve numerically this problem differs from the usual scheme only in the approximation of the first derivative of u with respect to t.

$$u_t(x,t) \approx (1-2\varepsilon)\frac{u_j^{n+1} - u_j^n}{\Delta t} + \varepsilon\frac{u_j^{n+1} - u_j^{n-1}}{\Delta t}. \tag{5}$$

In this way the solution, in faint differences, of the problem proposed by Maxwell-Cattaneo is given as follows:

$$\begin{aligned} u_j^{n+1} &= (1-2\varepsilon)\frac{u_j^{n+1} - u_j^n}{\Delta t} + \varepsilon\frac{u_j^{n+1} - u_j^{n-1}}{\Delta t} + \\ &+ \tau_0 \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta x^2}. \end{aligned} \tag{6}$$

It is important to note that for the NSFD solution it results in the usual scheme and for:

$$u_j^{n+1} = R(u_{j+1}^n + u_{j-1}^n) + (1-2R)u_j^n \tag{7}$$

on what $R = \Delta t / \Delta x^2$. It is possible to show that this scheme is stable for R = 1/4.

A. Energy-Based Numerical Analysis

Our main proposal in this work is to present an expression that determines the numerical energy of the NSFD method and then demonstrate that this system is dissipative.

Definition (Energy of NSFD scheme solutions):

Given that u_j^n is as discrete numerical solutions, as the sum of squares of u_j^n weighted by the product $\Delta x \Delta t$.

$$E = 2\Delta x\,\Delta t \sum_{j=0}^{J} (u_j^n)^2. \tag{8}$$

Proposition *(Energy Dissipation):* For all Δt and Δx the energy E is decreasing, ie:

$$E^{n+1} \le E^n, \quad \forall n \ge 0. \tag{9}$$

Proof:

Considering the definition of energy given earlier we have to

$$E^{n+1} - E^n = 2\Delta x\,\Delta t \sum_{j=0}^{J} (u_j^{n+1})^2 - (u_j^n)^2 \tag{10}$$

Consequently,

$$E^{n+1} - E^n = 2\Delta x\,\Delta t \sum_{j=0}^{J} (u_j^{n+1} + u_j^n).(u_j^{n+1} - u_j^n) \tag{11}$$

Considering the stability criterion R = 1/4 and replacing in the above expression we have:

$$E^{n+1} - E^n = \frac{1}{2}\Delta x\,\Delta t \sum_{j=0}^{J} (u_j^{n+1} + u_j^n).(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \tag{12}$$

Applying the distributive property of multiplication, we obtain a summation with three important plots. In the first plot we apply the homogeneous boundary conditions of Dirichlet [14]. In the second installment, we use inequality $ab \le \frac{a^2}{2} + \frac{b^2}{2}$. In the third installment we use the expression that defines the NSFD scheme. Thus, we arrive at the following result:

$$E^{n+1} - E^n \le \frac{1}{4}\left[-\Delta x\,\Delta t \sum_{j=0}^{J} (u_{j+1}^n + u_j^n)^2 + E^{n+1} - E^n\right] \tag{13}$$

Adding $\frac{1}{4}\Delta x\,\Delta t \sum_{j=0}^{J} (u_{j+1}^n + u_j^n)^2$ in the second member of inequality we have:

$$(1-\frac{1}{4})E^{n+1} \le (1-\frac{1}{4})E^n \tag{14}$$

Therefore,

$$E^{n+1} \le E^n. \tag{15}$$

III. COMPUTATIONAL SIMULATIONS

The computational simulations were done with the purpose of verifying the decay shown analytically in the proposed proposition. In this sense, we initially present a comparison between the exact solutions and the unusual scheme.

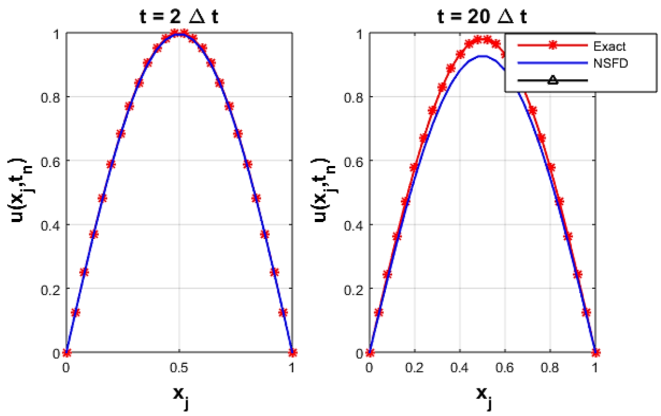

Figure 1. Comparison between the exact solution and the NSFD scheme for n = 2 and n = 20.

Figure 2 is a comparison between analytical and proposed energy in this article. The analytical energy used in this simulation was defined as follows:

$$E(t) = \frac{\tau_0}{2}\int_0^2 u_t^2 dt + \frac{1}{2}\int_0^2 u_x^2 dx \qquad (16)$$

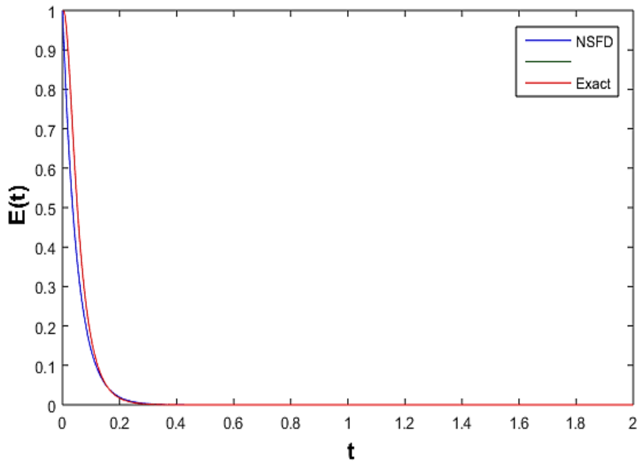

Figure 2. Comparison between the energies of the exact solution and the NSFD scheme.

Figure 3 represents a part of the behavior of the energies, especially the moment when the values of the energies intercept.

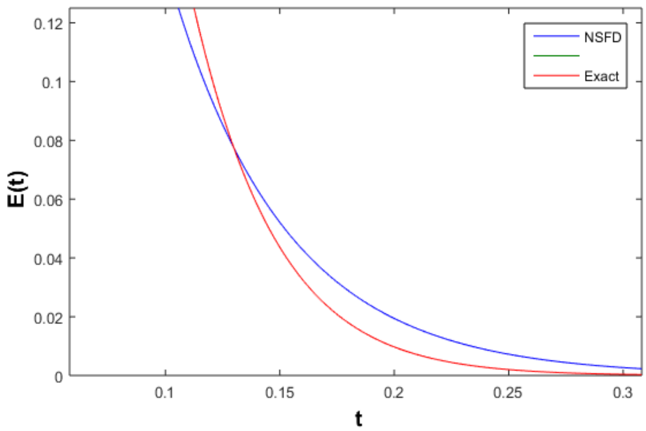

Figure 3. Momentum in which the energies of the exact solution and the NSFD scheme intersect.

IV. DISCUSSION AND CONCLUSIONS

Analyzing the solutions of differential equations obtained through numerical methods is a challenging task in computational mathematics and engineering. It is common to perform this analysis by graphically verifying the proximity between the analytical solution and the one obtained by the method (Figure 1). In figure 1, for example, we have two comparisons between the solutions at different times of time. In this figure we notice that for an instant of time n = 2 the solution of the scheme has a good approximation of the exact solution, but for n = 20 an approximation error is already perceptible.

An analysis based on comparisons of the solutions is mainly not feasible because it is not possible to verify the approximation at all simulation moments in a single moment. For this reason we present energy-based analysis in figure 2. In this graph we can summarize a comparison for all instants of simulation. We realize that solutions initially approach, but there is a distance over time.

In figure II, we also perceive that there is an intercession between the energies at a given instant. This intercession is shown in figure 3. This fact is important to mention because it is a further advantage of analyzing the method through energy. Figures 2 and 3 confirm what has been demonstrated analytically in the proposition we have demonstrated. That is, the energy of the NSFD scheme is decreasing and the heat conduction system is dissipative.

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lcineide Dutra Pessoa. Bachelor in science and technology and major in Civil Engineering at the Federal University of Maranhão.

Marta de Oliveira Barreiros. Bachelor in Information Systems from the Faculty of Imperatriz, Master in Biotechnology from the Federal University of Pará. PhD student in the Graduate Program in Electrical Engineering, Federal University of Maranhão.

Gean Carlos Lopes de Sousa, Graduated and Master in Mathematics from the Federal University of Pará, with experience in Numerical Analysis of Differential Equations. He is currently a professor at the Federal University of Maranhão and PhD student in the Graduate Program in Electrical Engineering, Federal University of Maranhão.