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On Soft $\sigma-$Algebra and Soft π–System

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Abstract-In this paper we consider the concepts of soft relation and soft mapping on soft sets. We represent some new results and theorems about these notions. Then we introduce the notions of soft $\sigma-$algebra, soft $\pi-$system and soft λ–system. We prove interesting theorems and give some examples about them.

Keywords- *Soft Set, Soft Relation, Soft Mapping, Soft π–System, Soft $\sigma-$Algebra*

I. INTRODUCTION

Soft set theory initiated by Molodtsov [10] in 1999 is one of the Mathematical concepts that has the parameterization tools deals with phenomena and concepts of life problems that contain uncertainties and vagueness arise in economics, engineering, mathematical modelling, environmental sciences and social sciences.

Molodtsov applied this theory in many fields like smoothness of functions, probability theory, measure theory game theory, Riemann integration etc.

Later Maji [8] defined several basic notions of soft set theory. Babitha and Sunil [1] introduced the soft set relations and many related concepts.

Wardowski in [14] established new concepts of soft element of a soft set and gave a natural definition of a soft mapping and discussed its properties.

In this paper we consider the concepts of soft relation and soft mapping according to the definitions of Wardowski. We represent new results and theorems about soft mapping. After that we introduce the notion of soft σ-algebras, soft π-system and soft $\lambda-$system over the universal set U and parameters set E .We prove important theorems and give some examples about them.

For more information about soft set theory, soft $\sigma-$algebra, Fuzzy soft set theory and some applications of them we refer to [2-7], [11-13].

II. PRELIMINARIES

Throughout this paper R (A) is the set of soft real number. Denote by U an initial universe, by E a set of parameters and by P (U) the collection of all subsets of U.

1) *Definition [10]:*

Let A be a nonempty subset of E. A soft set $(F,)$ on U is a set of the form $F_A = \{(p,\Lambda_F(p)){:}p \in E\}$, where $\Lambda_F{:}E \to P(U)$ is a set valued map such that $\Lambda_F(p) = \emptyset$ for $p \notin A$. Λ_F is called an approximate function of F_A. The elements of F_A of the form $(p,\emptyset)$ will be omitted. If a set of parameters A is of no importance, we will write F instead of F_A.

The collection of all soft sets on U will be denoted by (U).

2) *Example:*

Let $U = \{1,2,3,4,5,6,7,8,9,10\}$ and $E = \{even,\ odd,\ maltiple\ of\ 3\}$ and $A = \{even,\ odd\}$. We define on U a soft set F_A as follows:

$$F_A = \{(even, \{2,4,6,8,10\}), (odd, \{1,3,5,7,9\})\}.$$

3) *Definition [14]:*

An empty soft set, denoted by $\widetilde{\emptyset}$, is a soft set of the form $\{(p,\emptyset){:}p \in E\}$.

4) *Definition [14]:*

A soft set F_A is called the $A\_$ universal soft set and is denoted by $I_{\tilde{A}}$ if $\Lambda_F(p) = U$ for each $p \in A$, i.e., $I_{\tilde{A}} = \{(p,\ U){:}p \in A\}$.

5) *Example:*

Let $U = \{1,3,5,7,9,11,13\}$ and A = {positive, odd} then

F_A= {(positive, U), $(odd,\ U)$}= $I_{\tilde{A}}$.

6) *Definition [14]:*

Let $F_1, F_2 \in \mathrm{S(U)}$. F_1 is called a soft subset of F_2, which is denoted by $F_1 \tilde{\subseteq} F_2$, if $\Lambda_{F_1}(\mathrm{p}) \subseteq \Lambda_{F_2}(\mathrm{p})$ for each $p \in$ E. Obviously, for each $F \in \mathrm{S(U)}$, we have

$$\widetilde{\emptyset} \tilde{\subseteq} F, F \tilde{\subseteq} I_{\tilde{E}}.$$

7) Example:

Let $U = \{U_1, U_2, U_3\}$, E = { e_1, e_2}and let F_1, F_2 be of the form $F_1 = \{(e_1, \{U_1\}), (e_2, \{U_1, U_3\})\}$

, $F_2 = \{(e_1, \{U_1, U_2\}), (e_2, \{U\})\}$ then $F_1 \tilde{\subseteq} F_2$.

8) Definition [14]:

Let $F_1, F_2 \in$ S(U) . We say that the soft sets F_1, F_2 are equal, which is denoted by $F_1 = F_2$ if $\Lambda_{F_1}(\mathrm{p}) = \Lambda_{F_2}(\mathrm{p})$ for all $p \in$ E. It is clear that $F_1 = F_2$ if and only if $F_1 \tilde{\subseteq} F_2$ and $F_2 \tilde{\subseteq} F_1$.

9) Definition [14]:

Let $F_1, F_2 \in$ S(U) . We define a soft union $\tilde{\cup}$, a soft product $\tilde{\cap}$ and soft difference $\tilde{\setminus}$ of the soft sets as follows:

1. $F_1 \tilde{\cup} F_2 = \left\{\left(p, \Lambda_{F_1}(\mathrm{p}) \cup \Lambda_{F_2}(\mathrm{p})\right): \mathrm{p} \in \mathrm{E}\right\}$.

2. $F_1 \tilde{\cap} F_2 = \left\{\left(p, \Lambda_{F_1}(\mathrm{p}) \cap \Lambda_{F_2}(\mathrm{p})\right): \mathrm{p} \in \mathrm{E}\right\}$.

3. $F_1 \tilde{\setminus} F_2 = \left\{\left(p, \Lambda_{F_1}(\mathrm{p}) \setminus \Lambda_{F_2}(\mathrm{p})\right): \mathrm{p} \in \mathrm{E}\right\}$.

10) Example:

Let U = { U_1, U_2, U_3, U_4, U_5} and E = { e_1, e_2, e_3, e_4}. For the soft sets of the form

$F_1 = \{(e_1, \{U_1, U_2, U_3\}), (e_3, \{U_3, U_4, U_5\})\}$,

$F_2 = \{(e_3, \{U_1, U_3\}), (e_4, \{U_1, U_2, U_5\})\}$

we have:

$F_1 \tilde{\cup} F_2 = \{(e_1, \{U_1, U_2, U_3\}), (e_3, \{U_1, U_3, U_4, U_5\}), (e_4, \{U_1, U_2, U_5\})\}$.

$F_1 \tilde{\cap} F_2 = \{(e_3, \{U_3\})\}$.

$F_1 \tilde{\setminus} F_2 = \{(e_1, \{U_1, U_2, U_3\}), (e_3, \{U_4, U_5\})\}$.

11) Definition [14]:

Let $\{F_i\}_{i \in I} \tilde{\subseteq}$ S(U) . A generalized soft union $\tilde{\cup}$ and a soft product $\tilde{\cap}$ of the family of the soft sets $\{F_i\}_{i \in I}$ are defined as follows:

1. $\tilde{\cup}_{i \in I} F_i = \left\{\left(p, \bigcup_{i \in I} \Lambda_{F_i}(\mathrm{p})\right): \mathrm{p} \in \mathrm{E}\right\}$.

2. $\tilde{\cap}_{i \in I} F_i = \left\{\left(p, \bigcap_{i \in I} \Lambda_{F_i}(\mathrm{p})\right): \mathrm{p} \in \mathrm{E}\right\}$.

12) Definition [14]:

Let $F \in$ S(U). A soft complement of F, denoted by $F^{\tilde{C}}$ is a set of the form $F^{\tilde{C}} = \left\{\left(p, U \setminus \Lambda_F(\mathrm{p})\right): \mathrm{p} \in \mathrm{E}\right\}$. Clearly we have the following properties:

$$F^{\tilde{C}} = I_{\tilde{E}} \tilde{\setminus} F, \qquad \left(F^{\tilde{C}}\right)^{\tilde{C}} = F, \qquad \left(\tilde{\emptyset}\right)^{\tilde{C}} = I_{\tilde{E}}.$$

III. Soft elements of soft sets

1) Definition [14]:

Let $F \in$ S(U). We say that $\alpha = (p, \{u\})$ is a nonempty soft element of F if p $\in$ E and $u \in \Lambda_F(p)$. The pair $(p, \emptyset)$, where p $\in$ E, will be called an empty soft elements of F.

Nonempty soft elements of F and soft elements of F will be called the soft elements of F. The fact that α is a soft element of F will be denoted by $\alpha \tilde{\in} F$.

2) Proposition [14]:

Let $F_1, F_2, F \in$ S(U) . Then we have

1. $\forall$ p $\in$ E , $(\mathrm{p}, \emptyset) \tilde{\in} F$,

2. $\alpha \tilde{\in} F \Leftrightarrow \{\alpha\} \tilde{\subseteq} F$,

3. $\alpha \tilde{\in} F_1 \tilde{\cap} F_2 \Leftrightarrow \alpha \tilde{\in} F_1 \wedge \alpha \tilde{\in} F_2$,.

4. $\alpha \tilde{\in} F_1 \tilde{\cup} F_2 \Leftrightarrow \alpha \tilde{\in} F_1 \cup \alpha \tilde{\in} F_2$,.

5. $\alpha \tilde{\in} F_1 \tilde{\setminus} F_2 \Leftrightarrow \alpha \tilde{\in} F_1 \wedge \alpha \tilde{\notin} F_2$, for each nonempty soft element α.

3) Example:

Let U = { u_1, u_2, u_3, u_4} and E = { p_1, p_2, p_3}. Take a soft set $F \in$ S(U) of the form $F = \{(p_1, \{u_1, u_2, u_3\})\}$. Then all the soft elements of F are the following:

$(p_1, \emptyset), (p_2, \emptyset), (p_3, \emptyset), (p_2, \{u_1\}), (p_2, \{u_2\}), (p_2, \{u_3\})$.

4) Proposition [14]:

For each $F \in$ S(U), the following holds:

$F = \tilde{\cup}\left\{\{\alpha\}: \alpha \tilde{\in} F\right\}$.

5) Proposition [14]:

Let $F_1, F_2 \in$ S(U) . Then the following holds:

$F_1 \tilde{\subseteq} F_2 \Leftrightarrow \forall \alpha \left(\alpha \tilde{\in} F_1 \Rightarrow \alpha \tilde{\in} F_2\right)$.

IV. Soft mapping

1) Definition [14]:

Let $F_1, F_2 \in$ S(U) . The cartesian product of F_1, F_2, denoted by $F_1 \tilde{\times} F_2$, is a set on U × U of the form

$F_1 \tilde{\times} F_2 = \{(p_1, p_2), \Lambda_F(p_1) \times \Lambda_F(p_2): p_1, p_2 \in E\}$.

2) Example:

Let $U = \{u_1, u_2, u_3\}$ and E = { p_1, p_2}. We define $F_1, F_2 \in$ S(U) as follows:

$F_1 = \{(p_1, \{u_1, u_2\})\}$, $F_2 = \{(p_1, \{u_1\}), (p_3, \{u_1, u_2, u_3\})\}$.

Then the soft product of F_1, F_2 is of the form:

$F_1 \tilde{\times} F_2 = \{\left((p_1, p_1), \{u_1, u_2\} \times \{u_1\}\right), \left((p_1, p_2), \{u_1, u_2\} \times \emptyset\right)$,

$\left((p_1, p_3), \{u_1, u_2\} \times \{u_1, u_2, u_3\}\right), ((p_2, p_1), \emptyset \times \{u_1\}),$

$\left((p_2, p_2), \emptyset \times \emptyset\right), \left((p_2, p_3), \emptyset \times \{u_1, u_2, u_3\}\right), \left((p_3, p_1), \emptyset \times \{u_1\}\right),$

$\left((p_3, p_2), \emptyset \times \emptyset\right), \left((p_3, p_3), \emptyset \times \{u_1, u_2, u_3\}\right)\} =$

$\{\left((p_1, p_1), \{(u_1, u_1), (u_2, u_1)\}\right),$

$\left((p_1, p_3), \{(u_1, u_1), (u_1, u_2), (u_1, u_3), (u_2, u_1), (u_2, u_2), (u_2, u_3)\}\right)\}$.

3) Definition [14]:

Let $F_1, F_2 \in$ S(U) . A soft set R is called a soft relation from F_1 to F_2 if $\widetilde{\subseteq} F_1 \widetilde{\times} F_2$, i.e. , R is a set of the form

$$R = \left\{\left((p,q), u_p \times u_q\right): p, q \in E,\ u_p \subseteq \Lambda_{F_1}(\mathrm{p}), u_q \subseteq \Lambda_{F_2}(\mathrm{p})\right\}.$$

If $\left((p,q), u_p \times u_q\right) \in R$, then we will write $(p, u_p)R(q, u_q)$.

4) Example:

Let F_1, F_2 be as in example 4.2. Then

$$R = \{((p_1, p_1), \{(u_1, u_1), (u_2, u_1)\}), ((p_1, p_3), \{(u_2, u_3)\})\},$$

is an example of soft relation fromF_1 toF_2. Thus we can write

$(p_1, \{u_1, u_2\})R(p_1, \{u_1\})$ & $(p_1, \{u_2\})R(p_3, \{u_3\})$.

5) Definition [14]:

Let $G \in$ S(U) . A soft relation $T \widetilde{\subseteq} F \widetilde{\times} G$ is called a soft mapping from F to G, which is denoted by $T: F \widetilde{\rightarrow} G$, if the following two conditions are satisfied:

(SM1) For each soft element $\alpha \widetilde{\in} F$, there exists only one soft element $\beta \widetilde{\in} G$ such that $\alpha T \beta$ (which will be noted as $T(\alpha) = \beta$).

(SM2) For each empty soft element $\alpha \widetilde{\in} F$, $T(\alpha)$ is an empty soft element of G.

6) Remark [14]:

The above mentioned definition of soft mapping is different from the notion of soft function introduced by Babitha and Sunil in [1] and also differs from the concept of soft mapping by Kharal and Ahmad in [4] and by Majumdar and Samanta in [9]. This new approach enables one to obtain a natural behavior of soft mapping similar to classical mapping.

7) Example:

Let F_1, F_2 be as in example 4.2 and let $T \widetilde{\subseteq} F_1 \widetilde{\times} F_2$ be of the form

$$T = \{((p_2, p_2), \emptyset \times \emptyset), ((p_1, p_3), \{u_2\} \times \{u_1\}), ((p_1, p_1), \{u_1\} \times \{u_1\})\}.$$

Then T is a soft mapping from F to G and can be written as:

$T(p_1, \{u_2\}) = (p_3, \{u_1\})$, $T(p_1, \{u_1\}) = (p_1, \{u_1\})$, $T(p_2, \emptyset) = (p_2, \emptyset)$.

8) Definition [14]:

Let $F, G \in$ S(U) and let $T: F \widetilde{\rightarrow} G$ be a soft mapping. The image of $X \widetilde{\subseteq} F$ under soft mapping T is a soft set, denoted by $T(X)$, of the form

$$T(X) = \widetilde{\cup}_{\alpha \in X}\{T_\alpha\}.$$

It is clear that $T(\widetilde{\emptyset}) = \widetilde{\emptyset}$ for each soft mapping T.

9) Definition [14]:

Let $F, G \in$ S(U) and let $T: F \widetilde{\rightarrow} G$ be a soft mapping. The inverse of $Y \widetilde{\subseteq} G$ under soft mapping T is the soft set, denoted by $T^{-1}(Y)$, of the form

$$T^{-1}(Y) = \widetilde{\cup}\left\{\{\alpha\}: \alpha \in F, T(\alpha) \widetilde{\in} Y\right\}.$$

10) Remark [14]:

According to the condition (SM2) of Definition 4.5 the inverse under soft mapping T in Definition 4.9 is well defined in particular, the inverse of empty soft set under T is always a soft set of F.

11) Proposition [14]:

Let $F, G \in$ S(U) , $X, X_1, X_2 \widetilde{\subseteq} F$, $Y, Y_1, Y_2 \widetilde{\subseteq} G$ and let $T: F \widetilde{\rightarrow} G$ be a soft mapping. Then the following hold:

1. $X_1 \widetilde{\subseteq} X_2 \Rightarrow T(X_1) \subseteq T(X_2)$.
2. $Y_1 \widetilde{\subseteq} Y_2 \Rightarrow T^{-1}(Y_1) \subseteq T^{-1}(Y_2)$.
3. $X \subseteq T^{-1}(T(X))$.
4. $T(T^{-1}(Y)) \subseteq Y$.
5. $T(X_1 \widetilde{\cup} X_2) = T(X_1) \widetilde{\cup} T(X_2)$.
6. $T(X_1 \widetilde{\cap} X_2) \widetilde{\subseteq} T(X_1) \widetilde{\cap} T(X_2)$.
7. $T^{-1}(Y_1 \widetilde{\cup} Y_2) = T^{-1}(Y_1) \widetilde{\cup} T^{-1}(Y_2)$.
8. $T^{-1}(Y_1 \widetilde{\cap} Y_2) = T^{-1}(Y_1) \widetilde{\cap} T^{-1}(Y_2)$.

12) Definition:

Let $F, G \in$ S(U) and let $T: F \widetilde{\rightarrow} G$ be a soft mapping. Then T is called an injective soft mapping if for each soft elements α_1, α_2 with $\alpha_1 \neq \alpha_2$ we have $T(\alpha_1) \neq T(\alpha_2)$.

13) Definition:

Let $F, G \in$ S(U) and let $T: F \widetilde{\rightarrow} G$ be a soft mapping. Then T is called a surjective soft mapping if $(F) = G$, i.e. , for each soft element $\beta \widetilde{\in} G$ there exists one soft element $\alpha \widetilde{\in} F$ such that $T(\alpha) = \beta$.

Soft mapping $T: F \widetilde{\rightarrow} G$ is called bijective soft mapping if it is injective and surjective soft mapping.

14) Definition:

Let $F, G \in$ S(U) and let $T: F \widetilde{\rightarrow} G$ be a soft mapping. Then T is called constant soft mapping if for every soft element $\alpha \widetilde{\in} F$, $T(\alpha)$ has the same soft element $k \widetilde{\in} G$.

15) Definition:

Let $F \in$ S(U). Then the soft mapping $I: F \widetilde{\rightarrow} F$ is called identity soft mapping if for each soft element $\alpha \widetilde{\in} F$ we have $I(\alpha) = \alpha$.

16) Definition:

Let $F, G \in$ S(U) and let $T: F \widetilde{\rightarrow} G$ be a soft mapping. Then the relation

$$T^{-1} = \{\left((p_j, p_i), \Lambda_G(p_j) \times \Lambda_F(p_i)\right): ((p_i, p_j), \Lambda_G(p_i) \times \Lambda_F(p_j)) \in T\}$$

is called the soft inversion mapping of T.

17) Definition:

Let $F, G, H \in S(U)$ and let $T_1: F \rightrightarrows G$, $T_2: G \rightrightarrows H$ be two soft mappings. We define the composition of T_1 and T_2 and denote it by $T_2 \circ T_1$ as follows:

$(T_2 \circ T_1)(\alpha) = T_2(T_1(\alpha))$, $\alpha \tilde{\in} F$.

18) Theorem:

Let $F, G \in S(U)$ and let $T: F \rightrightarrows G$ be an injective soft mapping. Then the inversion soft mapping T^{-1} is injective. If $T: F \rightrightarrows G$ is surjective then T^{-1} is also surjective mapping.

Proof:

Suppose for soft elements $\beta_1, \beta_2 \tilde{\in} G$ we have $\beta_1 \neq \beta_2$. We want to prove that $T^{-1}(\beta_1) \neq T^{-1}(\beta_2)$.

Let $T^{-1}(\beta_1) = \alpha_1$ & $T^{-1}(\beta_2) = \alpha_2$ for some soft elements $\alpha_1, \alpha_2 \tilde{\in} F$. Then we have $T(\alpha_1) = \beta_1$ & $T(\alpha_2) = \beta_2$.

Thus $T(\alpha_1) \neq T(\alpha_2)$. Since T is injective then $\alpha_1 \neq \alpha_2$. Consequently

$T^{-1}(\beta_2) = \alpha_2 \neq T^{-1}(\beta_1) = \alpha_1$. Therefore T^{-1} is injective. (1)

Now let $\alpha \tilde{\in} F$. Since T is surjective then there exists unique soft element $\beta \tilde{\in} G$ such that $T(\alpha) = \beta$. Thus for $\alpha \tilde{\in} F$ we have $\alpha = T^{-1}(\beta)$.

Hence T^{-1} is surjective. (2)

From (1) & (2) we deduce that T^{-1} is bijective soft mapping.

19) Theorem:

Let $F, G, H \in S(U)$ and let $T_1: F \rightrightarrows G$, $T_2: G \rightrightarrows H$ be two surjective soft mappings. Then the composition soft mapping $T_2 \circ T_1: F \rightrightarrows H$ is surjective and we have $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

Proof:

Suppose $\alpha_1, \alpha_2 \tilde{\in} G$ be two different soft elements then $\alpha_1 \neq \alpha_2$. Since T is injective then $T_1(\alpha_1) \neq T_2(\alpha_2)$. Thus $T_2(T_1(\alpha_1)) \neq T_2(T_1(\alpha_2))$ as T_2 is injective. Consequently $T_2 \circ T_1(\alpha_1) \neq T_2 \circ T_1(\alpha_2)$. This proves that $T_2 \circ T_1$ is injective.

Now suppose that $\gamma \tilde{\in} H$ (be a soft element). Since T_2 is surjective then there exists a soft element $\beta \tilde{\in} G$ such that $T_2(\beta) = \gamma$. Hence there exists $\alpha \tilde{\in} F$ such that $T_2(\alpha) = \beta$ as T_1 is surjective. Therefore for every $\gamma \tilde{\in} H$ there exists $\alpha \tilde{\in} F$ such that $T_2(T_1(\alpha)) = T_2(\beta) = \gamma$.

In the other hand $(T_2 \circ T_1)(\alpha) = \gamma$. This proves that $T_2 \circ T_1$ is surjective. Therefore $T_2 \circ T_1$ is bijective soft mapping. Since $T_1, T_2, T_2 \circ T_1$ are bijective soft mapping then we can easily prove that $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

V. Soft σ – algebra

1) Definition [3]:

A collection M of subsets of X is called algebra of sets if:

1. $A \cup B$ is in M whenever A and B are in M.

2. A^C is in M whenever A is in M.

2) Definition [3]:

A collection J of subsets of X is called a sigma algebra of sets if:

1. $X \in J$.

2. $A^C \in J$ when $A \in J$.

3. Countable union of subsets in J is also in J.

3) Definition:

Let $\mathcal{m}$ is a collection of soft sets over the universal set U and parameters set E. We say that $\mathcal{m}$ is a soft σ – algebra if :

1. $I_{\tilde{E}} \in \mathcal{m}$.

2. $F_i^{\tilde{c}} \in \mathcal{m}$ when $F_i \in \mathcal{m}$.

3. $\cup_i F_i \in \mathcal{m}$ whenever $F_i \in \mathcal{m}$ for $i = 1,2, \ldots$.

If $\mathcal{m}$ is a soft σ – algebra then $(U, \mathcal{m})$ is called soft measurable space and members of $\mathcal{m}$ are soft measurable.

4) Definition:

Let $F, G \in S(U)$ and let $T: F \rightrightarrows G$ is a soft mapping. Then T is called soft measurable mapping if for every measurable soft set $O \tilde{\subseteq} G$, $T^{-1}(O)$ is soft measurable in F.

5) Example:

Let U is a universal set and E be a set of parameters. Then the collection $\mathcal{m}_1 = \{\tilde{\emptyset}, I_{\tilde{E}}\}$ is a soft σ – algebra.

6) Example:

Let U is a universal set and E be a set of parameters. Let $\mathcal{m}_2$ be the collection of all soft set over U (which can be defined from E to $P(U)$).

Then $\mathcal{m}_2$ is a soft σ – algebra. If $\mathcal{m}$ is any other soft σ – algebra then we have:

$\mathcal{m}_1 \tilde{\subseteq} \mathcal{m} \tilde{\subseteq} \mathcal{m}_2$.

7) Example:

Let $U = \{p_1, p_2, p_3, p_4\}$ and $E = \{e_1, e_2\}$. Assume that

$\mathcal{m} = \{\tilde{\emptyset}, \{(e_1, \{p_1\})\}, \{(e_1, \{p_2, p_3, p_4\})\}, \{(e_1, \{p_1, p_2, p_3, p_4\})\},$
$\{(e_2, \{p_1, p_2, p_3, p_4\})\}\}$.

Then we can see that $\mathcal{m}$ is a soft – algebra.

8) Definition:

Let $\mathcal{m}$ is a soft – algebra over universal set U with parameters set E. Then we have the following:

1) $\tilde{\emptyset} \in \mathcal{m}$.

2) If $F_{A_1}, \dots, F_{A_n} \in m$, then $F_{A_1} \tilde{\cup} \dots \tilde{\cup}\ F_{A_n} \in m$.

3) If $F_{A_1}, \dots, F_{A_n} \in m$, then $F_{A_1} \tilde{\cap} \dots \tilde{\cap}\ F_{A_n} \in m$.

4) Let $\{F_{A_1}, F_{A_2}, \dots\}$ is a collection of soft sets in m, then $\tilde{\cap}_i F_{A_i} \in m$.

5) If $F_A, F_B \tilde{\in} m$ then $F_A \tilde{\setminus} F_B \in m$.

Proof:

1) Since $I_{\tilde{E}} \in m$, thus $I_{\tilde{E}}{}^C = \tilde{\emptyset} \in m$.

2) We have

$F_{A_1} \tilde{\cup} \dots \tilde{\cup}\ F_{A_n} = F_{A_1} \tilde{\cup} \dots \tilde{\cup}\ F_{A_n} \tilde{\cup} \tilde{\emptyset} \tilde{\cup} \tilde{\emptyset} \tilde{\cup} \dots .$

Since m is closed under countable union thus $F_{A_1} \tilde{\cup} \dots \tilde{\cup}\ F_{A_n} \in m$.

3) Since $F_{A_1}, \dots, F_{A_n} \in m$, So we have $F_{A_1}{}^{\tilde{c}}, \dots, F_{A_n}{}^{\tilde{c}} \in m$. Therefore we can write:

$F_{A_1} \tilde{\cap} \dots \tilde{\cap}\ F_{A_n} = (F_{A_1}{}^{\tilde{c}} \tilde{\cup} \dots \tilde{\cup}\ F_{A_n}{}^{\tilde{c}})^{\tilde{c}} \in m,$

as m is closed under soft complement.

4) We know $\tilde{\cap}_{n=1} F_{A_n} = (\underset{n=1}{\tilde{\cup}} F_{A_n}{}^{\tilde{c}})^{\tilde{c}} \in m$, as m is closed under soft countable unions and soft complement.

5) Let $F_A, F_B \in m$. Then $F_B{}^{\tilde{c}} \tilde{\in} m$. Hence we have $F_A \setminus F_B = F_A \cap F_B{}^{\tilde{c}} \in m$.

9) Theorem:

Let m_1, m_2 are two soft σ – algebras. Then the soft intersection of m_1, m_2 also is soft σ – algebra.

Proof

It can be easily proved by definition of soft σ_algebras.

10) Remark [5]:

Soft union of two soft σ – algebras may not be soft σ – algebra.

11) Definition [5]:

Let U is a universal set and E be a set of parameters. Suppose that $\mathcal{N}$ is a nonempty collection of soft sets over U. Then we define the smallest soft σ – algebra that contains all soft sets of $\mathcal{N}$ the soft σ – algebra generated by $\mathcal{N}$ and is denoted by $\sigma(\mathcal{N})$.

12) Theorem [5]:

Let $\mathcal{N}$ be the collection of soft subsets over U. Then there is a smallest soft σ – algebra containing $\mathcal{N}$.

13) Definition:

Let U is a universal set and E be a set of parameters. We say that a collection ρ of soft sets over U is soft π – system if:

1) $\tilde{\emptyset} \in \rho$.

2) $F_A \cap F_B \in \rho$ whenever $F_A, F_B \in \rho$.

14) Definition:

Let U is a universal set and E be a set of parameters. We say that a collection $\mathcal{L}$ of soft sets over U is soft λ – system if:

1) $I_{\tilde{E}} \in \mathcal{L}$.

2) $F_A{}^{\tilde{c}} \in \mathcal{L}$ whenever $F_A \in \mathcal{L}$.

3) If $F_{A_n} \in \mathcal{L}, n \geq 1$ with $F_{A_i} \tilde{\cap} F_{A_j} = \emptyset\ (\forall i \neq j)$, then $\underset{n=1}{\tilde{\cup}} A_n \in \mathcal{L}$.

15) Example:

Let $U = \{ p_1, p_2, p_3, p_4\}$ and E = $\{ e_1, e_2, e_3\}$. Assume that

$$\rho = \{\tilde{\emptyset}, \{(e_1, \{p_1, p_2\})\}, \{(e_2, \{p_1, p_2, p_3\})\}, \{(e_3, \{p_1, p_2, p_4\})\}, \{(e_1, \{p_1\}), (e_3, \{p_1, p_2, p_4\})\}, \{(e_1, \{p_1\}), \{(e_3, \{p_2, p_4\})\}\}.$$

Then we can see that ρ is a soft π – system.

16) Example:

Let $U = \{p_1, p_2, p_3, p_4\}$ and E = $\{ e_1, e_2, e_3, e_4\}$. Assume that

$$\mathcal{L} = \{\tilde{\emptyset}, I_{\tilde{E}}, \{(e_1, \{p_1, p_2\})\}, \{(e_2, \{p_1, p_4\})\}, \{(e_3, \{p_2, p_3\})\}, \{(e_4, \{p_3, p_4\})\}\}.$$

Then $\mathcal{L}$ is a soft λ – system but is not a soft σ – algebra. Because $\mathcal{L}$ is not closed under unions (but is closed under disjoint unions).

17) Theorem:

1) A soft σ – algebra is a soft π – system.

2) A soft σ – algebra is a soft λ – system.

Proof:

1) Let U is a universal set and E be a set of parameters. Suppose that the collection m of soft sets over U is a soft σ – algebra. Then $I_{\tilde{E}} \in m$. Thus $\tilde{\emptyset} = (I_{\tilde{E}})^{\tilde{c}} \in m$.

Also let $F_A, F_B \in m$. Then $F_A \cup F_B \in m$. Hence as we know $F_A{}^{\tilde{c}}, F_B{}^{\tilde{c}} \in m$. Hence $F_A \tilde{\cap}\ F_B = (F_A{}^{\tilde{c}} \tilde{\cup}\ F_B{}^{\tilde{c}})^{\tilde{c}} \in m$. Consequently m is a soft π – system.

2) By similar manner we can prove that m is a soft λ – system.

18) Remark:

A soft σ – algebra is a soft λ – system but a soft λ – system is not a soft σ – algebra. Because a soft λ – system is closed under soft disjoint unions not soft unions. (see Example 5.16.)

19) Theorem:

Let U is a universal set and E be a set of parameters. Suppose that the collection m of soft sets over U is soft π – system and soft λ – system. Then m is a soft σ – algebra.

Proof:

Let m is a collection of soft sets over U which is soft π – system and soft λ – system. According to the definitions of soft π – system and soft λ – system we can see that:

1) $I_{\tilde{E}} \in m$.

2) $F_A{}^{\tilde{c}} \in m$ whenever $F_A \in m$.

Thus we need to prove that m is closed under soft unions.

Let $F_{A_1}, \dots, F_{A_n} \in m$ which that soft sets F_{A_i} are not disjoint. We want to prove that $\tilde{\cup}_{i=1} F_{A_i} \in m$.

Suppose $F_{B_1}, \dots, F_{B_n}$ are some soft sets over U which we construct by the following manner:

$F_{B_1} = F_{A_1}, F_{B_2} = F_{A_2} \tilde{\setminus} F_{A_1}, F_{B_3} = F_{A_3} \tilde{\setminus} (F_{A_1} \tilde{\cup} F_{A_2}), \dots, F_{B_n} = F_{A_n} \tilde{\setminus} \tilde{\cup}_{i=1} F_{A_i}$.

Therefore we have $F_{B_n} = F_{A_n} \tilde{\cap}_{i=1} (\tilde{\cup}_{i=1} F_{A_i}.)^{\tilde{c}}$. Since $\mathcal{m}$ is a soft π – system and soft λ – system thus $F_{B_n} \in \mathcal{m}$. Also we have $\tilde{\cup}_{i=1} F_{A_i} = \tilde{\cup}_{i=1} F_{B_i}$.

According to the above mentioned statements $\{F_{B_i}\}$ are soft disjoint. Since $\mathcal{m}$ is soft λ – system thus $\tilde{\cup}_{i=1} F_{B_i} \in \mathcal{m}$. Consequently $\tilde{\cup}_{i=1} F_{A_i} \in \mathcal{m}$. This proves that $\mathcal{m}$ is a soft σ – algebra.

20) Theorem:
Let U is a universal set and E be a set of parameters. Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ are two soft λ – system over U. Then $\mathcal{L}_1 \tilde{\cap} \mathcal{L}_2$ is a soft λ – system.

Proof:
Can be prove similar to Theorem 5.9.

21) Definition:
Let U is a universal set and E be a set of parameters. We define the soft λ – system generated by the collection $\mathcal{L}$ of soft sets over which is denoted by $\sigma(\mathcal{L})$ to be the intersection of all soft λ – system containing $\mathcal{L}$. One can show that $\sigma(\mathcal{L})$ is the smallest soft λ – system that contains $\mathcal{L}$.

22) Lemma:
Let $\mathcal{L}$ is a collection of soft sets over universal set U and parameter set E and $\sigma(\mathcal{L})$ is the smallest soft λ – system that contains $\mathcal{L}$. Suppose that

$\Lambda_1 = \{F_A \in \sigma(\mathcal{L}) | \forall F_B \in \mathcal{L}, F_A \tilde{\cap} F_B \in \sigma(\mathcal{L})\}$.

Then Λ_1 is a soft λ – system that contains $\mathcal{L}$.

Proof:
1) $I_{\tilde{E}} \in \Lambda_1$, since for every $F_B \in \mathcal{L} \subseteq \sigma(\mathcal{L})$, we have $I_{\tilde{E}} \tilde{\cap} F_B = F_B \in \sigma(\mathcal{L})$.

2) Let $F_A \in \Lambda_1$. We want to show that ${F_A}^C \in \Lambda_1$.

We have ${F_A}^C \in \sigma(\mathcal{L})$ as $F_A \in \sigma(\mathcal{L})$ (since $F_A \in \Lambda_1$). Let $F_B \in \mathcal{L}$ is arbitrary so $F_B \in \sigma(\mathcal{L})$. Thus ${F_A}^C \tilde{\cap} F_B \in \sigma(\mathcal{L})$. Consequently ${F_A}^C \in \Lambda_1$.

3) Let $F_{A_i} \in \Lambda_1$ with $F_{A_i} \tilde{\cap} F_j = \emptyset$ (for $i \neq j$). We want to show that $\tilde{\cup}_{i=1} F_{A_i} \in \Lambda_1$. Pick $F_B \in \mathcal{L}$ then $F_B \cap F_{A_n} \in \sigma(\mathcal{L})$ (since$F_B \in \mathcal{L} \subseteq \sigma(\mathcal{L})$ & $F_{A_n} \in \sigma(\mathcal{L})$).

Thus $(\tilde{\cup}_{n=1} F_{A_n}) \tilde{\cap} F_B = \tilde{\cup}_{n=1} (F_{A_n} \tilde{\cap} F_B) \in \sigma(\mathcal{L})$ (since $\sigma(\mathcal{L})$ is soft λ – system). Consequently$\tilde{\cup}_{n=1} F_{A_n} \in \Lambda_1$.

Therefore Λ_1 is a soft λ – system, because $\sigma(\mathcal{L})$ is the smallest soft λ – system that contains $\mathcal{L}$. Hence we have $\sigma(\mathcal{L}) \subseteq \Lambda_1$. By definition of Λ_1 we see that $\Lambda_1 \subseteq \sigma(\mathcal{L})$. Thus two sets are equal, $\sigma(\mathcal{L}) = \Lambda_1$.

23) Lemma:
Let U is a universal set and E be a set of parameters. According to the statements of last lemma the set

$\Lambda_2 = \{F_A \in \sigma(\mathcal{L}) | \forall F_B \in \sigma(\mathcal{L}), F_A \tilde{\cap} F_B \in \sigma(\mathcal{L})\}$

is a soft λ – system.

Proof:
It can be proved by similar method which we stated for last lemma.

24) Remark:
By the above mentioned result we have $\mathcal{L} \subseteq \Lambda_2$, since the intersection of any two soft sets from $\sigma(\mathcal{L})$ and $\mathcal{L}$ lies in $\sigma(\mathcal{L})$.

Therefore $\sigma(\mathcal{L}) \subseteq \Lambda_2$. Also we have $\Lambda_2 \subseteq \sigma(\mathcal{L})$ (by definition). So the two sets are equal.

Consequently $\sigma(\mathcal{L})$ is closed under intersection with soft sets from $\mathcal{L}$.

25) Theorem:
Let $\mathcal{L}$ is a collection of soft sets over universal set U and parameter set E. If $\mathcal{L}$ is a soft π – system then $\sigma(\mathcal{L})$ is a soft π – system.

Proof:
It follows from lemma 5.22, 5.23 and remark 5.24.

26) Theorem:
Let $\mathcal{L}$ is a collection of soft sets over universal set U and parameter set E. Then $\sigma(\mathcal{L})$ is equal with the σ – algebra generated by $\mathcal{L}$.

Proof:
We saw that $\sigma(\mathcal{L})$ is a soft λ – system. By theorem 5.25 $\sigma(\mathcal{L})$ is also a soft π – system. Thus by theorem 5.19, $\sigma(\mathcal{L})$ is a soft σ – algebra containing $\mathcal{L}$. So the σ – algebra generated by the collection $\mathcal{L}$ is soft subset of $\sigma(\mathcal{L})$.

Similarly since every soft σ – algebra is a soft λ – system thus $\sigma(\mathcal{L})$ is soft subset of the σ – algebra generated by $\mathcal{L}$. Therefore $\sigma(\mathcal{L})$ is equal by the σ – algebra generated by $\mathcal{L}$.

27) Corollary:
Let ρ is a soft π – system over universal set U and parameter set E and $\mathcal{L}$ is a soft λ – system over U. Then the soft σ – algebra generated by ρ is a soft subset of $\mathcal{L}$.

Proof:
According to theorem 5.26 the soft σ – algebra generated by ρ is equal with $\sigma(\mathcal{L}) \subseteq \mathcal{L}$.

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