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Homotopy Perturbation Method and Elzaki Transform for Solving Korteweg-DeVries (KDV) Equation

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Abstract-In this paper, a combined form of the Elzaki transform method with the homotopy perturbation method is proposed to solve Korteweg-DeVries (KDV) Equation. This method is called Elzaki transform homotopy perturbation method (ETHPM). The proposed method was derived by combining Elzaki transform and homotopy perturbation method. This method was found to be more efficient and easy to solve linear and nonlinear differential equations.The (ETHPM) finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. The results reveal that the proposed method is very efficient, simple and can be applied to other nonlinear problems.

Keywords- *Elzaki Transform, Homotopy Perturbation Method, He's Polynomials, Korteweg-DeVries (KDV) Equation*

I. INTRODUCTION

In the recent years, the idea of homotopy was coupled with perturbation. The fundamental work was done by He In 1992. He [5–18] developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation for solving various physical problems .Many authors have applied this method successfully to problems arising in mathematics engineering. The KDV equation plays an important role in diverse areas of engineering and scientific applications, and therefore, enormous amount of research work has been invested in the study of KDV equations [29–33]. ELzaki transform is a useful technique for solving linear differential equations but this transform is totally incapable of handling nonlinear equations [3] because of the difficulties that are caused by the nonlinear terms. This paper is using homotopy perturbation method to decompose the nonlinear term, so that the solution can be obtained by iteration procedure. This means that we can use both ELzaki transform and homotopy perturbation methods to solve many nonlinear problems. The main aim of this paper is to consider the effectiveness of the Elzaki transform homotopy perturbation method in solving KDV Equations. In this paper we present some basic definitions of Elzaki transform and homotopy perturbation, also we present a reliable combination of homotopy perturbation method and Elzaki transform to obtain the solution of Korteweg-DeVries (KDV) Equation. The KDV equation can be presented in the following form

$$u_t - 6uu_x + u_{xxx} = 0$$

where $u\,(x,t)$ is the displacement.

The purpose of this paper is to extend the(HPTM)for the solution of Korteweg-DeVries (KDV) Equation. The method has been successfully applied for obtaining exact solutions for nonlinear equations.

II. BASIC IDEA

To illustrate the basic idea of this method, we consider a general nonlinear non homogeneous partial differential equation with initial conditions of the form

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \quad (1)$$

$$u(x,0) = h(x), u_t\,(x,0) = f\,(x)$$

Where D is the second order linear differential operator$D = \partial^2/\partial t^2$, is the linear differential operator of less order than D, N represents the general non-linear differential operator and $g(x,t)$ is the source term.

Taking Elzaki transform (denoted throughout this paper by E) on both sides of Eq. (1), to get

$$E[Du(x,t)] + E[Ru(x,t)] + E\,[Nu(x,t)] = E\,[g(x,t)] \quad (2)$$

Using the differentiation property of Elzaki transform and above initial conditions, we have

$$E[u(x,t)] = v^2E[g(x,t)] + v^2h(x) + v^3f(x) - v^2E[Ru(x,t) + Nu(x,t)] \quad (3)$$

Operating with the Elzaki inverse on both sides of Eq.(3) gives

$$u(x,t) = G(x,t) - E^{-1}\left[v^2\,E[Ru(x,t) + Nu(x,t)]\right] \quad (4)$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial condition.

Now, we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (5)$$

And the nonlinear term can be decomposed as

$$\mathrm{Nu}(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (6)$$

Where $H_n(u)$ are He's polynomial and given by

$$H_n(u_0,\dots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \quad n = 0,1,2,3\dots \quad (7)$$

Substituting Eqs. (6) and (5) in Eq. (4) we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(E^{-1}\left[v^2 E\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \quad (8)$$

Which is the coupling of the Elzaki transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained

$p^0:\ u_0(x,t) = G(x,t),$

$p^1:\ u_1(x,t) = -E^{-1}\left[v^2 E[Ru_0(x,t) + H_0(u)]\right],$

$p^2:\ u_2(x,t) = -E^{-1}\left[v^2 E[Ru_1(x,t) + H_1(u)]\right],$

$p^3:\ u_3(x,t) = -E^{-1}\left[v^2 E[Ru_2(x,t) + H_2(u)]\right],$

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Then the solution is

$$u(x,t) = \lim_{p\to 1} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \quad (9)$$

III. Applications

In this section, the effectiveness and the usefulness of homotopy perturbation transform method (HPTM) are demonstrated by finding exact solutions of Korteweg-DeVries (KDV) Equation.

Example 1. Consider the following homogeneous KDV equation;

$$u_t - 6uu_x + u_{xxx} = 0 \quad (10)$$

With the initial condition;

$$u(x,0) = 6x \quad (11)$$

Applying the Elzaki transform on both sides of Eq. (10)

$$E[u_t] = E[6uu_x - u_{xxx}] \quad (12)$$

Using the differential property of Elzaki transform Eq. (12) can be written as

$$\frac{1}{v}E[u(x,t)] - vu(x,0) = E[6uu_x - u_{xxx}] \quad (13)$$

Using initial condition (11), Eq. (13) can be written as

$$E[u(x,t)] = v^2(6x) + v\,E[6uu_x - u_{xxx}] \quad (14)$$

The inverse Elzaki transform implies that:

$$u(x,t) = 6x - E^{-1}\left[v\,E[u_{xxx} - 6uu_x]\right] \quad (15)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 6x - PE^{-1}\left[v\,E\left(\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xxx} - \sum_{n=0}^{\infty} p^n H_n(u)\right)\right] \quad (16)$$

Where $H_n(u)$ are He's polynomials that represents the nonlinear terms.

The first few components of He's polynomials, are given by;

$$H_0(u) = u_0 u_{0x}$$

$$H_1(u) = u_0 u_{1x} + u_1 u_{0x} \quad (17)$$

$$H_2(u) = u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$p^0:\ u_0(x,t) = 6x$

$p^1:\ u_1(x,t) = -E^{-1}\left[v\,E[(u_0)_{xxx} - 6H_0(u)]\right] = 6^3 xt,$

$p^2:\ u_2(x,t) = -E^{-1}\left[v\,E[(u_1)_{xxx} - 6H_1(u)]\right] = 6^5 xt^2,$ (18)

$p^3:\ u_3(x,t) = -E^{-1}\left[v\,E[(u_2)_{xxx} - 6H_2(u)]\right] = 6^7 xt^3,$

Proceeding in a similar manner, we have

$p^4:\ u_4(x,t) = -E^{-1}\left[v\,E[(u_3)_{xxx} - 6H_3(u)]\right] = 6^9 xt^4,$

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Therefore the solution $u(x,t)$ is given by:

$$u(x,t) = 6x(1 + (36t) + (36t)^2 + (36t)^3 + (36t)^4 + \cdots) \quad (19)$$

in series form, and,

$$u(x,t) = \frac{6x}{1-36t}, \quad |36t| < 1 \quad (20)$$

in closed form.

Example 2. Consider the following homogeneous KDV equation;

$$u_t + 6uu_x + u_{xxx} = 0 \quad (21)$$

With the initial condition;

$$u(x,0) = x \quad (22)$$

Applying the Elzaki transform on both sides of Eq. (21)

$$E[u_t] = -E[6uu_x + u_{xxx}] \quad (23)$$

Using the differential property of Elzaki transform Eq. (23) can be written as

$$\frac{1}{v}E[u(x,t)] - vu(x,0) = -E[6uu_x + u_{xxx}] \quad (24)$$

Using initial condition (22), Eq. (24) can be written as

$$E[u(x,t)] = v^2(x) - v\,E[6uu_x + u_{xxx}] \quad (25)$$

The inverse Elzaki transform implies that:

$$u(x,t) = x - E^{-1}\left[\text{v}\, E[\mathrm{u_{xxx}} + 6\mathrm{uu_x}]\right] \quad (26)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} \mathrm{p^n u_n}(x,\mathrm{t}) = x - PE^{-1}[\text{v}\, E((\sum_{n=0}^{\infty} \mathrm{p^n u_n}(x,\mathrm{t}))_{xxx} + \sum_{n=0}^{\infty} \mathrm{p^n H_n(u)})] \quad (27)$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$\mathrm{p^0}:\ \mathrm{u_0}(x,\mathrm{t}) = x$$

$$\mathrm{p^1}:\ \mathrm{u_1}(x,\mathrm{t}) = -E^{-1}\left[\text{v}\, E[(\mathrm{u_0})_{xxx} - 6\mathrm{H_0(u)}]\right] = -x(6t),$$

$$\mathrm{p^2}:\ \mathrm{u_2}(x,\mathrm{t}) = -E^{-1}\left[\text{v}\, E[(\mathrm{u_1})_{xxx} - 6\mathrm{H_1(u)}]\right] = x(6t)^2, \quad (28)$$

$$\mathrm{p^3}:\ \mathrm{u_3}(x,\mathrm{t}) = -E^{-1}\left[\text{v}\, E[(\mathrm{u_2})_{xxx} - 6\mathrm{H_2(u)}]\right] = -x(6t)^3,$$

Proceeding in a similar manner, we have

$$\mathrm{p^4}:\ \mathrm{u_4}(x,\mathrm{t}) = -E^{-1}\left[\text{v}\, E[(\mathrm{u_3})_{xxx} - 6\mathrm{H_3(u)}]\right] = x(6t)^4,$$

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Therefore the solution $\mathrm{u}(x,\mathrm{t})$ is given by:

$$\mathrm{u}(x,\mathrm{t}) = x(1 - (6t) + (6t)^2 - (6t)^3 + (6t)^4 - (6t)^5 + \cdots) \quad (29)$$

in series form, and,

$$\mathrm{u}(x,\mathrm{t}) = \frac{x}{1+6t} \quad (30)$$

in closed form.

Example 3. Consider the following homogeneous KDV equation;

$$\mathrm{u_t} - 6\mathrm{uu_x} + \mathrm{u_{xxx}} = 0 \quad (31)$$

With the initial condition;

$$u(x,0) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} \quad (32)$$

Applying the Elzaki transform on both sides of Eq. (31)

$$E[u_t] = E[6\mathrm{uu_x} - \mathrm{u_{xxx}}] \quad (33)$$

Using the differential property of Elzaki transform Eq. (33) can be written as

$$\frac{1}{\text{v}}\mathrm{E}[\mathrm{u}(x,\mathrm{t})] - \text{v}\mathrm{u}(\mathrm{x},0) = E[6\mathrm{uu_x} - \mathrm{u_{xxx}}] \quad (34)$$

Using initial condition (32), Eq. (34) can be written as

$$\mathrm{E}[\mathrm{u}(x,\mathrm{t})] = \text{v}^2(-2\frac{k^2 e^{kx}}{(1+e^{kx})^2}) + \text{v}\, E[6\mathrm{uu_x} - \mathrm{u_{xxx}}] \quad (35)$$

The inverse Elzaki transform implies that:

$$u(x,t) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} - E^{-1}\left[\text{v}\, E[\mathrm{u_{xxx}} - 6\mathrm{uu_x}]\right] \quad (36)$$

Now, we apply the homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} \mathrm{p^n u_n}(x,\mathrm{t}) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} - PE^{-1}[\text{v}\, E((\sum_{n=0}^{\infty} \mathrm{p^n u_n}(x,\mathrm{t}))_{xxx} - \sum_{n=0}^{\infty} \mathrm{p^n H_n(u)})] \quad (37)$$

Comparing the coefficient of like powers of p, the following approximations are obtained;

$$\mathrm{p^0}:\ \mathrm{u_0}(x,\mathrm{t}) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2}$$

$$\mathrm{p^1}:\ \mathrm{u_1}(x,\mathrm{t}) = -E^{-1}\left[\text{v}\, E[(\mathrm{u_0})_{xxx} - 6\mathrm{H_0(u)}]\right] = -2\frac{k^5 e^{kx}(e^{kx}-1)}{(1+e^{kx})^3}t, \quad (38)$$

$$\mathrm{p^2}:\ \mathrm{u_2}(x,\mathrm{t}) = -E^{-1}\left[\text{v}\, E[(\mathrm{u_1})_{xxx} - 6\mathrm{H_1(u)}]\right] = -\frac{k^8 e^{kx}(e^{2kx}-4e^{kx}+1)}{(1+e^{kx})^4}t^2,$$

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Therefore, the solution $\mathrm{u}(x,\mathrm{t})$ is given by:

$$\mathrm{u}(x,\mathrm{t}) = -2\frac{k^2 e^{kx}}{(1+e^{kx})^2} -2\frac{k^5 e^{kx}(e^{kx}-1)}{(1+e^{kx})^3}t - \frac{k^8 e^{kx}(e^{2kx}-4e^{kx}+1)}{(1+e^{kx})^4}t^2 + \cdots \quad (39)$$

Using Taylor series, the closed form solution will be as follows:

$$\mathrm{u}(x,\mathrm{t}) = -2\frac{k^2 e^{k(x-k^2 t)}}{\left(1+e^{k(x-k^2 t)}\right)^2} \quad (40)$$

IV. CONCLUSIONS

In this paper, we have applied the homotopy perturbation method and Elzaki transform to Korteweg-DeVries (KDV) Equation. It can be concluded that the ETHPM is a very powerful and efficient technique in finding exact and approximate solutions for nonlinear problems.By using this method we obtain a new efficient recurrent relation to solve (KDV) Equation. In conclusion, ETHPM provide highly accurate numerical solutions for nonlinear problems in comparison with other method. The results show that the ETHPM is a powerful mathematical tool for solving the KDV having wide applications in engineering and applied mathematics.

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