International Journal of
Science and Engineering Investigations vol. 10, issue 110, March 2021
Received on March 13, 2021 ISSN: 2251-8843

Analytical Solutions of Diffusive Lotka-Volterra System taking into account the chemical and biological interpretations

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Abstract-Lotka-Volterra equations are among the most commonly used equations in chemistry, biochemistry, ecology, biology, medicine, economics etc., and they are mainly solved by numerical methods; analytical methods subject to specific conditions in non-diffusion conditions are present. However, advanced concepts of Riemannian geometry including Lie symmetries and its generalizations (conditional Q-symmetries in this study) leads to the geometric classification of the analytic solution of the diffusive Lotka-Volterra system that is a lot more complicated than the analytical solutions resulting from non-diffusive setting. Accordingly, the exact solution of the diffusive Lotka-Volterra system along with its chemical and biological interpretations is presented for a nonlinear model of the competition of two populations. A comprehensive geometric classification of the exact solution of these equations is provided. Also, the solution for the non-linear Neumann boundary value problem and its different results for the definitive and soft competition model are carefully expressed.

Keywords- *Diffusive Lotka-Volterra System, Q-conditional Symmetry, Von Neumann Boundary Value Problem, Soft Competition, Definitive Competition*

I. Introduction

Lotka and Volterra were pioneering researchers who set the background for the ecological mathematics. In the 1920s, they developed a mathematical model for prey-predator interaction. The first example of oscillation in a homogeneous chemical reaction is observed in a research text in 1921, [1]. The chemical interpretation was then assigned this observation to the Lotka-Volterra model in the sense that the oscillations in the cold flame can be described using this model.

Numerous attempts have been made to numerically solve these equations, [2]. One can even point to the use of approximate inertial manifolds to solve these equations numerically, [3]. On the other hand, there are also attempts to find exact inertial manifolds for differential equations and dynamical systems, [4-5]. Therefore, it seems that advanced geometrical concepts can provide comprehensive and precise results to provide analytical solutions to the Lotka-Volterra system.

The analytical solutions of the Lotka-Volterra model for sustained chemical oscillation were explicitly investigated by Evan, [6-7], which resulted in a simple modification of the model using the second-order nonlinear differential equation.

The Diffusive Lotka-Volterra system is a much more sophisticated model than the non-diffusive one whose comprehensive study began about 40 years ago [8-10]. Today, the Diffusive Lotka - Volterra System, or DLVS, is used as one of the most widely used equations in modelling the interactive systems for a variety of processes in chemistry, biochemistry, ecology, biology, medicine, economics and more [11-16].

This paper presents the analytical solutions of these commonly used equations by advanced geometric concepts including generalized Lie symmetries (Q -conditional symmetries) method. Using the Q-conditional symmetries, the DLVS is reduced to a set of ordinary differential equations (ODEs). Examples of possible exact solutions and possible biological interpretations of the obtained results are demonstrated; in particular, those who represent different scenarios of competition between the two populations are thoroughly discussed.

In Section 2, the geometrical concepts necessary for solving the diffusive Lotka-Volterra equation are presented. Section 3 introduces the diffusive Lotka-Volterra system and the results of its conditional symmetries are presented in Section 4. The results of reducing the diffusive Lotka-Volterra system to a system of ODEs and their exact solutions are investigated in Section 5. To express the biological interpretation, the studies of plane wave solutions for DLVS have been made for the model of the competition between two populations. Also, the solution of the Neumann boundary value problem (BVP) for the nonlinear DLVS, and its various interpretations in the cases of definitive/soft competition and their results are comprehensively presented.

II. Geometric Concepts

As the standard notions of differential geometry and Lie groups ([17-18]) an n -dimensional manifold is a set M , together with countable coordinate charts $U_\alpha \subset M$ and one-to-one local coordinate maps $\phi_\alpha : U_\alpha \to V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbf{R}^m$, which satisfy the following properties:

The coordinated charts cover M : $\bigcup_\alpha U_\alpha = M$.

On the overlap of any pair of coordinate charts $U_\alpha \cap U_\beta$ the composite map $\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha\left(U_\alpha \cap U_\beta\right) \to \chi_\beta\left(U_\alpha \cap U_\beta\right)$ is a smooth function.

If $x \in U_\alpha$, $\tilde{x} \in U_\beta$ are distinct points of M, then, there exist open subsets $W \subset V_\alpha$, $\tilde{W} \subset V_\beta$ with $\chi_\alpha(x) \in W$, $\chi_\beta(\tilde{x}) \in \tilde{W}$ and $\chi_\alpha^{-1}(W) \cap \chi_\beta^{-1}(\tilde{W}) \neq \phi$.

A local r-parameter Lie group consists of open subsets $V_0 \subset V \subset \mathbf{R}^r$ containing origin, and the smooth maps $m : V \times V \to \mathbf{R}^r$ and $i : V_0 \to V$ defining the action group and the inversion action (respectively) which satisfy the following properties:

associativity: $m(x, m(y,z)) = m(m(x,y), z)$ for all $x,y,z \in V$ with $m(x,y), m(y,z) \in V$,

identity: $m(0,x) = x = m(x,0)$ for all $x \in V$,

inversion: $m(x, i(x)) = 0 = m(i(x), x)$ for all $x \in V_0$.

A local group of transformations acting on a manifold M is given by a (local) Lie group G, an open subset U as the domain of definition of the group action with $\{e\} \times M \subset U \subset G \times M$, and a smooth map $\Psi : U \to M$ which satisfy the following properties:

If $(h,x) \in U$, $(g, \Psi(h,x)) \in U$, and $(g.h, x) \in U$ then $\Psi(g, \Psi(h,x)) = \Psi(g.h, x)$.

$\Psi(e,x) = x$ for all $x \in M$.

If $(g,x) \in U$ then $(g, \Psi(h,x)) \in U$ and $\Psi\left(g^{-1}, \Psi(g,x)\right) = x$.

For brevity, $\Psi(g,x)$ is shown as $g.x$.

At each point of a smooth parametrized curve $\gamma : I \to M$ of a subinterval of $\mathbf{R}$ on a manifold M there is a tangent vector $\dot{\gamma}(t) = \frac{d\gamma}{dt} = \left(\dot{\gamma}^1(t), ..., \dot{\gamma}^n(t)\right)$.

For an n-dimensional manifold M, $TM|_x$ which is a collection of all tangent vectors to all possible curves passing through a given point x in M forms the is an n-dimensional vector space, with $\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}\right\}$ as a basis for it.

A vector field v on M associates the tangent vector $v|_x \in TM|_x$ to any point $x \in M$ that $v|_x$ varies smoothly of each point to the other. In local coordinates $(x_1, ..., x_n)$, it is in the form $v|_x = \xi^1(x)\frac{\partial}{\partial x_1} + \xi^2(x)\frac{\partial}{\partial x_2} + ... + \xi^n(x)\frac{\partial}{\partial x_n}$ where each $\xi^i(x)$ is a smooth function of x.

The maximal integral parametrized curves $\dot{\gamma}(t) = v|_{\gamma(t)}$ passing through $x \in M$ is shown by $\Psi(t,x)$, and is called the flow generated by a vector field v or a one-parameter group of transformations which leads to the interpretation of the vector field v as the infinitesimal generator of the action Ψ. Also, $\frac{d}{dt}\Psi(t,x) = v|_{\Psi(t,x)}$. This flow is denoted by $\exp(tv)x \equiv \Psi(t,x)$, then $\frac{d}{dt}\left[\exp(tv)x\right] = v|_{\exp(tv)x}$ for all $x \in M$.

There is a one-to-one correspondence between the local one-parameter groups of transformations and their infinitesimal generators, i.e., for the vector field $v = \sum \xi^i(x)\frac{\partial}{\partial x_i}$ on M and the smooth function $f : M \to \mathbf{R}$, $\frac{d}{dt} f\left(\exp(tv)x\right)|_{t=0} = v(f)(x)$.

If v and w are vector fields on M, then their Lie bracket $[v,w]$ is the unique vector field satisfying $[v,w](f) = v(w(f)) - w(v(f))$ for all smooth functions $f : M \to \mathbf{R}$.

For any group element g of a Lie group G, the right multiplication map $R_g : G \to G$ defined by $R_g(h) = h.g$ is a diffeomorphism, with inverse $R_{g^{-1}} = \left(R_g\right)^{-1}$. A vector field v on G is called right-invariant if $dR_g(v|_h) = v|_{R_g(h)} = v|_{hg}$ for all g and h in G. The set of all right-invariant vector fields forms a vector space.

A Lie algebra is a vector space $\mathbf{G}$ with a bilinear operation $[.,.] : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$, called the Lie bracket for $\mathbf{G}$, satisfying the following axioms: $[cv + c'v, w] = c[v,w] + c'[v', w]$ and $[v, +cw + c'w'] = c[v,w] + c'[v, w']$, for all $c, c' \in \mathbf{R}$, $[v,w] = -[w,v]$ and $[u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0$, for all u, v, v', w, w' in $\mathbf{G}$.

The flow generated by a right-invariant vector field $v \neq 0$ through the identity, namely $g_t = \exp(tv)e \equiv \exp(tv)$ is defined for all $t \in \mathbf{R}$ and forms a one-parameter subgroup of G. Conversely, any connected one-dimensional subgroup of G is generated by such a right-invariant vector field in the above manner.

For a smooth real-valued function $f(x) = f(x_1, ..., x_r)$ of r independent variables, there are $r_k \equiv \binom{r+k-1}{k}$, different k-th order partial derivatives of f. The multi–index notation $\partial f(x) = \partial^k f(x) / \partial x_{j_1} \partial x_{j_2} ... \partial x_{j_k}$ is used to these derivatives which $J = (j_1, ..., j_k)$, $1 \leq j_i \leq p$, $1 \leq i \leq k$ Specifies the type of derivative adopted.

More generally, if $f: X \to U$ is a smooth function from $X \approx \mathbf{R}^r$ to $U \approx \mathbf{R}^s$, so $u = f(x) = (f_1(x),..., f_s(x))$ has $s.r_k$ numbers $u_J^\alpha = \partial_J f_\alpha(x)$ needed to represent all the different k -th order derivatives of the components of f at a point x. A typical point in $U^{(n)} = U \times U_1 \times ... \times U_n$ will be denoted by $u^{(n)}$ which indicates all the derivatives of functions $u = f(x)$ of all orders from 0 to n.

A system of n -th order differential equations in r independent and s dependent variables is defined as $\Delta_\rho(x,u^n) = 0$, for $\rho = 1,...,l$, involving $x = (x_1,...,x_r)$, $u = (u_1,...,u_s)$ and the derivatives of u with respect to x up to order n. For smooth function $\Delta\left(x,u^{(n)}\right) = \left(\Delta_1\left(x,u^{(n)}\right),...,\Delta_l\left(x,u^{(n)}\right)\right)$, Δ ; $\Delta: X \times U^{(n)} \to \mathbf{R}$ can be considered as a smooth map of the jet space $X \times U^{(n)}$ into an l -dimensional Euclidean space. For more details, refer to [17-18].

Consider the k -order PDE

$$L(t,x,u,u_1,...,u_k) = 0, \ k \geq 1, \tag{1}$$

where $u = u(t,x)$ is an unknown function, u_s is a totality of s-order derivatives of $u(t,x)$, $s = 1,...,k$, and L is a given smooth function.

Definition 1. ([12]) Operator Q which satisfied in the equation

$$\begin{gathered} Q = \xi^0(t,x,u)\partial_t + \xi^1(t,x,u)\partial_x + \eta(t,x,u)\partial_u, \\ \left(\xi^0\right)^2 + \left(\xi^1\right)^2 \neq 0, \end{gathered} \tag{2}$$

is called the Q-conditional symmetry of PDE (1) if the following invariance criterion $Q_k(L)|_M = 0$ is satisfied where the differential operator Q_k is the k -order prolongation of operator (2) and the manifold M is defined by the system of equations $L = 0$, $Q(U) = 0$, $\frac{\partial^{p+q} Q(u)}{\partial t_p x_q} = 0$, $1 \leq p+q \leq k-1$ in the prolonged space of the variables $t,x,u,u_1,...,u_k$. In this case, $u = u(t,x)$ is called the Q -conditional invariant under the operator Q.

III. DIFFUSIVE LOTKA-VOLTERRA SYSTEM

The Lotka-Volterra system consists of two nonlinear ordinary differential equations (ODEs) of the form

$$\begin{aligned} \frac{du_e}{dt} &= u_e(a - bu_d), \\ \frac{du_d}{dt} &= u_d(-c + du_e), \end{aligned} \tag{3}$$

where the functions $u_e(t)$ and $u_d(t)$ describe the time evolution of the numbers of prey and predators, respectively, a,b,c and d are positive parameters with well-known interpretation, [19-20].

The system (3) with quadratic nonlinearities can also describe other types of interaction (competition, mutualism), hence the classical Lotka– Volterra model is usually presented in the form

$$\begin{aligned} \frac{du_e}{dt} &= u_e(a_1 + b_1u_d + c_1u_d), \\ \frac{du_d}{dt} &= u_d(a_2 + b_2u_e + c_2u_d), \end{aligned} \tag{4}$$

Depending on the signs of coefficients in (4), three common types of interaction between two populations arise, namely: predator–prey interaction, competition and mutualism.

IV. SYMMETRIES OF DIFFUSIVE LOTKA-VOLTERRA SYSTEM

The two-component diffusive Lotka–Volterra system

$$\begin{aligned} \lambda_1\frac{du_e}{dt} &= (u_e)_{xx} + u_e(a_1 + b_1u_e + c_1u_d), \\ \lambda_2\frac{du_d}{dt} &= (u_d)_{xx} + u_d(a_2 + b_2u_e + c_2u_d), \end{aligned} \tag{5}$$

is examined, where $\lambda_k > 0$, a_k, b_k and c_k $(k = 1,2)$ are arbitrary constants, $u_e = u_e(t,x)$ and $u_d = u_d(t,x)$ are unknown functions presenting, for example, the population concentrations.

Throughout the article it is assumed that system (5) is nonlinear and $b_2^2 + c_1^2 \neq 0$, i.e., the system cannot consist of two independent equations.

It is obvious that DLVS (5) with arbitrary coefficients is invariant under the two-dimensional Lie algebra generated by the operators of translation $P_t = \partial_t$ and $P_x = \partial_x$ one may look for the plane wave solutions of DLVS (5) of the form

$$u_e = \varphi(v), \ u_d = \psi(v), \ v = x - \alpha t, \ \alpha \in \mathbf{R} \tag{6}$$

According to the literature ([13]), DLVS (5) is invariant concerning three- and higher-dimensional Lie algebra if and only if its reaction terms and the corresponding symmetry operator(s) have the forms listed in Table 1.

If DLVS (5) with other reaction terms admits a nontrivial Lie algebra, then it is reduced to one of the forms presented in Table 1 by a local substitution from the set $u_e \to c_{11}\exp(c_{10}t)u_e + c_{12}$, $u_d \to c_{21} + c_{22}\exp(c_{20}t)u_d$, where c_{ki} $(k = 1,2, \ i = 0,1,2)$ are some correctly-specified constants.

TABLE I. LIE SYMMETRIES OF DLVS (5)

Number	Reaction terms	Restriction	Lie symmetries extending algebra $P_t = \partial_t$ and $P_x = \partial_x$
1	$u_e(b_1u_e + c_1u_e), u_d(b_2u_e + c_2u_d)$		$D = 2tP_t + xP_x - 2\left(u_e\partial_{u_e} + u_d\partial_{u_d}\right)$
2	$b_1u_e^2, b_2u_eu_d$		$D, u_d\partial_{u_d}$
3	$b_1u_e^2, 0$		$D, u_d\partial_{u_d}, X^\infty = P(t,x)\partial_{u_d}$
4	$u_e(a_1 + b_1u_e), b_2u_eu_d$		$u_d\partial_{u_d}$
5	$u_e(a_1 + b_1u_e), 0$		$u_d\partial_{u_d}, X^\infty = P(t,x)\partial_{u_d}$
6	$u_e(a_1 + b_1u_e), u_d(a_1 + b_1u_e)$	$\lambda_1 = \lambda_2$	$u_d\partial u_d, u_e\partial u_d$
7	$b_1u_e^2, b_1u_eu_d$	$\lambda_1 = \lambda_2$	$u_d\partial_{u_d}, u_e\partial_{u_d}, D\ R = b_1tu_e\partial_{u_e} + \partial_{u_d}$

V. MAIN RESULT

Theorem 1. DLVS (5) admits Q -conditional symmetry under the relevant coefficient restrictions.

Proof. The system of determining equations (DEs) for finding the Q -conditional symmetry operator are listed in Table 2.

The differential consequences of the equations (4) and (5) of Table 2 concerning the variables of u_e and u_d lead to the expressions $(\lambda_1 + \lambda_2)\xi_{u_d}^2 = 0$ and $(\lambda_1 + \lambda_2)\xi_{u_e}^2 = 0$, so that $\xi_{u_e} = \xi_{u_d} = 0$. Having $\xi = \xi(t,x)$, equations (1)-(5) of Table 2 can be easily solved and one arrives at

$$\eta^1 = q^1(t,x)u_d + r^1(t,x)u_e + p^1(t,x),$$
$$\eta^2 = q^2(t,x)u_e + r^2(t,x)u_d + p^2(t,x), \qquad (7)$$

hence, the most general form of symmetry operators for system (5) is

$$Q = \partial_t + \xi\partial_x + (q^1u_d + r^1u_e + p^1)\partial_{u_e}$$
$$+(q^2u_e + r^2u_d + p^2)\partial_{u_d}, \qquad (8)$$

where the functions q^k, r^k and p^k $(k = 1,2)$ should be found from the remaining equations (6)-(11) of Table 2.

Substituting (7) and $\xi_{u_e} = \xi_{u_d} = 0$ into equations (6)-(11) of Table 2, those can be split with respect to u_e, u_d, u_e^2, u_d^2 and u_eu_d. Finally, one obtains the system listed in Table 3 which does not involve the unknown functions u_e and u_d.

TABLE II. DETERMINED EQUATIONS FOR Q -CONDITIONAL SYMMETRIES OF DLVS (5)

Equation	Number
$\xi_{u_eu_e} = \xi_{u_du_d} = \xi_{u_eu_d} = \eta^1_{u_du_d} = \eta^2_{u_eu_e} = 0$	1
$2\lambda_1\xi\xi_{u_e} + \eta^1_{u_eu_e} - 2\xi_{xu_e} = 0$	2
$2\lambda_2\xi\xi_{u_d} + \eta^2_{u_du_d} - 2\xi_{xu_d} = 0$	3
$(\lambda_1 + \lambda_2)\xi\xi_{u_d} + 2\eta^1_{u_eu_d} - 2\xi_{xu_d} = 0$	4
$(\lambda_1 + \lambda_2)\xi\xi_{u_e} + 2\eta^2_{u_eu_d} - 2\xi_{xu_e} = 0$	5
$(\lambda_1 - \lambda_2)\xi\eta^1_{u_d} + 2\eta^2_{xu_d} + 2u(a_1 + b_1u_e + c_1u_d)\xi_{u_d} - 2\lambda_1\xi_{u_d}\eta^1 = 0$	6
$(\lambda_2 - \lambda_1)\xi\eta^2_{u_e} + 2\eta^2_{xu_e} + 2u_d(a_2 + b_2u_e + c_2u_d)\xi_{u_e} - 2\lambda_2\xi_{u_e}\eta^2 = 0$	7
$\lambda_1\left(2\xi_{u_e}\eta^1 - \xi_t - \xi_{u_d}\eta^2 - 2\xi\xi_x\right) + \lambda_2\xi_{u_d}\eta^2 - 3\xi_{u_e}u_e(a_1 + b_1u_e + c_1u_d) - \xi_{u_d}u_d(a_2 + b_2u_e + c_2u_d) - 2\eta^1_{xu_e} + \xi_{xx} = 0$	8
$\lambda_2\left(2\xi_{u_d}\eta^2 - \xi_t - \xi_{u_e}\eta^1 - 2\xi\xi_x\right) + \lambda_1\xi_{u_e}\eta^1 - 3\xi_{u_d}u_d(a_2 + b_2u_e + c_2u_d) - \xi_{u_e}u_e(a_1 + b_1u_e + c_1u_d) - 2\eta^1_{xu_d} + \xi_{xx} = 0$	9
$\lambda_1\left(\eta^1_t + \eta^2\eta^1_{u_d} + 2\xi_x\eta^1\right) - \lambda_2\eta^2\eta^1_{u_d} - \eta_1(a_1 + 2b_1u_e + c_1u_d) - c_1\eta_2u_e + u_e(a_1 + b_1u_e + c_1u_d)\left(\eta^1_{u_e} - 2\xi_x\right) + \eta^1_{u_d}u_d(a_2 + b_2u_e + c_2u_d) = \eta^1_{xx}$	10
$\lambda_2\left(\eta^2_t + \eta^2\eta^2_{u_e} + 2\xi_x\eta^2\right) - \lambda_1\eta^1\eta^2_{u_e} - \eta_2(a_2 + b_2u_e + 2c_2u_d) - b_2\eta_1u_d + \eta^2_{u_e}u_e(a_1 + b_1u_e + c_1u_d) + u_d(a_2 + b_2u_e + c_2u_d)\left(\eta^2_{u_d} - 2\xi_x\right) = \eta^2_{xx}$	11

Analysis of the system of the equations (1)-(14) of Table 3 shows that its solutions essentially depends on the relation between λ_1 and λ_2. So, In the case $\lambda_1 \neq \lambda_2$, DLVS (5) is Q-conditionally invariant under operator (8) if and only if $b_1 = b_2 = b$ and $c_1 = c_2 = c$.

TABLE III. SIMPLIFICATIONS OF THE EQUATIONS OF TABLE 2.

Equation	Number
$(c_1 - c_2)q^1 = 0$	1
$(b_1 - b_2)q^2 = 0$	2
$c_1 q^2 + b_1\left(r^1 + 2\xi_x\right) = 0$	3
$b_2 q^1 + c_2\left(r^2 + 2\xi_x\right) = 0$	4
$(2b_1 - b_2)q^1 + c_1(r^2 + 2\xi_x) = 0$	5
$(2c_2 - c_1)q^2 + b_2\left(r^1 + 2\xi_x\right) = 0$	6
$(\lambda_1 - \lambda_2)\xi q^1 + 2q^1_x = 0$	7
$(\lambda_2 - \lambda_1)\xi q^2 + 2q^2_x = 0$	8
$\lambda_1(\xi_t + 2\xi\xi_x) + 2r^1_x - \xi_{xx} = 0$	9
$\lambda_2(\xi_t + 2\xi\xi_x) + 2r^2_x - \xi_{xx} = 0$	10
$\lambda_1\left(r^1_t + 2r^1\xi_x\right) + (\lambda_1 - \lambda_2)q^1q^2 - c_1p^2 - 2b_1p^1 - 2a_1\xi_x - r^1_{xx} = 0$	11
$\lambda_2\left(r^2_t + 2r^2\xi_x\right) + (\lambda_2 - \lambda_1)q^1q^2 - b_2p^1 - 2c_2p^2 - 2a_2\xi_x - r^2_{xx} = 0$	12
$\lambda_1\left(q^1_t + 2q^1\xi_x\right) + (\lambda_1 - \lambda_2)q^1r^2 - (a_1 - a_2)q^1 - c_1p^1 - q^1_{xx} = 0$	13
$\lambda_2(q^2_t + 2q^2\xi_x) + (\lambda_2 - \lambda_1) + (a_1 - a_2)q^2 - b_2p^2 - q^2_{xx} = 0$	14
$\lambda_1\left(p^1_t + 2p^1\xi_x\right) + (\lambda_1 - \lambda_2)q^1p^2 - a_1p^1 - p^1_{xx} = 0$	15
$\lambda_2\left(p^2_t + 2p^2\xi_x\right) + (\lambda_2 - \lambda_1)q^2p - a_2p^2 - p^2_{xx} = 0$	14

If $bc = 0$ and $b^2 + c^2 \neq 0$, then system (5) and the Q-conditional symmetries have the forms (9)

$$\lambda_1(u_e)_t = (u_e)_{xx} + u_e(a_1 + u_e),$$
$$\lambda_2(u_d)_t = (u_d)_{xx} + u_d u_e,$$
$$Q = \partial_t + \frac{2\alpha_1}{\lambda_1 - \lambda_2}\partial_x + \left(\phi(t)\mathrm{e}^{\alpha_1 x}u_e + \mathrm{e}^{\alpha_1 x}\right. \tag{9}$$
$$\left.\left(\lambda_2\phi'(t) + a_1\phi(t) - \alpha_1^2\phi(t)\right) + \alpha_2 u_d\right)\partial_{u_d},$$

(up to local transformations $u_e \to bu_e$, $u_d \to \exp\left(\frac{a_2}{\lambda_2}t\right)u_d$, $cu_d \to u_e$ and $u_e \to \exp\left(\frac{a_1}{\lambda_1}t\right)u_d$) where the function $\phi(t) \neq 0$ is the general solution of the linear ODE $\lambda_2^2\phi'' + \lambda_2\left(a_1 - 2\alpha_1^2\right)\phi' + \alpha_1^2\left(\alpha_1^2 - a_1\right)\phi = 0$.

If $bc \neq 0$ and the additional restrictions $q^1_x = q^2_x = 0$ hold, then exactly three cases (up to local transformations $u_e \to bu_e$, $cu_d \to u_d$, $u_e \to u_d$ and $u_d \to u_e$) exist when system (5) admits Q-conditional symmetry operators. They are listed as Table 4.

TABLE IV. CASES WHICH DLVS (5) ADMITS Q-CONDITIONAL SYMMETRY OPERATORS FOR $bc \neq 0$ IN TABLE 3

Equations	Case
$\lambda_1(u_e)_t = (u_e)_{xx} + u_e(a_1 + u_e + u_d)$, $\lambda_2(u_d)_t = (u_d)_{xx} + u_d(a_2 + u_e + u_d), a_1 \neq a_2$, $Q_1 = (\lambda_1 - \lambda_2)\partial_t - (a_1u_d + a_2u_e + a_1a_2)\left(\partial_{u_e} - \partial_{u_d}\right), a_1a_2 \neq 0$, $Q_2 = (\lambda_1 - \lambda_2)\partial_t + (a_1 - a_2)u_e\left(\partial_{u_e} - \partial_{u_d}\right)$, $Q_3 = (\lambda_1 - \lambda_2)\partial_t - (a_1 - a_2)u_d\left(\partial_{u_e} - \partial_{u_d}\right)$	1
$\lambda_1(u_e)_t = (u_e)_{xx} + u_e(a + u_e + u_d)$, $\lambda_2(u_d)_t = (u_d)_{xx} + u_d(a + u_e + u_d)$, $Q_1 = (\lambda_1 - \lambda_2)\partial_t - a(u_d + u_e + a)\left(\partial_{u_e} - \partial_{u_d}\right), a \neq 0$, $Q_2 = (\lambda_1 - \lambda_2)t\partial_t - (\lambda_1u_d + \lambda_2u_e)\left(\partial_{u_e} - \partial_{u_d}\right), a \neq 0$	2
$\lambda_1(u_e)_t = (u_e)_{xx} + u_e(a\lambda_1 + u_e + u_d)$, $\lambda_2(u_d)_t = (u_d)_{xx} + u_d(a\lambda_2 + u_e + u_d), \quad a \neq 0$, $Q_1 = (\lambda_1 - \lambda_2)\partial_t - a(\lambda_1u_d + \lambda_2u_e + a\lambda_1\lambda_2)\left(\partial_{u_e} - \partial_{u_d}\right)$, $Q_2 = \partial_t + au_e\left(\partial_{u_e} - \partial_{u_d}\right)$, $Q_3 = \partial_t - au_d\left(\partial_{u_e} - \partial_{u_d}\right)$, $Q_4 = \left(e^{-at} - \alpha(\lambda_1 - \lambda_2)\right)\partial_t + a\alpha(\lambda_1u_d + \lambda_2u_e + a\lambda_1\lambda_2)\left(\partial_{u_e} - \partial_{u_d}\right), \alpha \neq 0$	3

In the case $\lambda_1 = \lambda_2$, DLVS (5) admits only such operators of the form (8),which are equivalent to the Lie symmetry operators, [11].

Theorem 2. Analytical solution of diffusive Lotka-Volterra system (5) can be obtained by considering the asymptotic and non-asymptotic behaviour of the response in accordance with the chemical and biological interpretations of DLVS (5).

Proof. The DLVS (5) is invariant under time and space translations, so, its arbitrary solution $(u_e)_0(t,x)$ and $(u_d)_0(t,x)$ generates a two-parameter family of solutions of the form $u_e(t - t_0, x - x_0)$ and $u_d(t - t_0, x - x_0)$. Therefore, these transitions can always be applied in order to get $t_0 = x_0 = 0$ in the solutions obtained below.

The plane wave solutions of DLVS (5) in the case when the system models the competition between two populations will be found.

Introducing new notation $\lambda_k \to \frac{1}{\lambda_k}$, $a_k \to \frac{a_k}{\lambda_k}$, $b_k \to -\frac{b_k}{\lambda_k}$ and $c_k \to -\frac{c_k}{\lambda_k}$, equation (5) is rewritten in the typical form

$$\begin{aligned}(u_e)_t &= \lambda_1 (u_e)_{xx} + u_e (a_1 - b_1 u_e - c_1 u_d),\\ (u_d)_t &= \lambda_2 (u_d)_{xx} + u_d (a_2 - b_2 u_e - c_2 u_d),\end{aligned} \tag{10}$$

where all the coefficients are positive. Using ansatz (6), the competition model (10) is reduced to ODEs

$$\begin{aligned}\lambda_1 \phi_{vv} + \alpha\phi_v + \phi(a_1 - b_1\phi - c_1\psi) = 0,\\ \lambda_2 \psi_{vv} + \alpha\psi_v + \phi(a_2 - b_2\phi - c_2\psi) = 0,\end{aligned} \tag{11}$$

For obtaining the solution of system of nonlinear ODEs (11), it is assumed that the following condition is satisfied:

$$\psi = \beta_0 + \beta_1\phi, \tag{12}$$

where β_0 and β_1 are certain constants to be found as below. Note that substituting (12) into (11) leads to an overdetermined system, which can possess non-constant solutions only under the condition $\lambda_1 = \lambda_2 = \lambda$. So, without loss of generality we set $\lambda = 1$. As a result, the single second-order ODE

$$\phi_{vv} + \alpha\phi_v + \phi(a - b\phi) = 0, \tag{13}$$

is obtained, where the parameters a and b depend essentially on the additional parameter β_0, namely:

$$a = \begin{cases} a_1 = a_2, & \beta_0 = 0,\\ a_1 - a_2\frac{c_1}{c_2}, & \beta_0 = \frac{a_2}{c_2},\end{cases} \qquad b = \begin{cases} \frac{c_1 b_2 - b_1 c_2}{c_1 - c_2}, & \beta_0 = 0,\\ b_1 + c_1\beta_1, & \beta_0 = \frac{a_2}{c_2},\end{cases} \tag{14}$$

$$\beta_1 = \begin{cases} \frac{b_1 - b_2}{c_2 - c_1}, & c_1 \ne c_2, \quad b_1 \ne b_2,\\ -\frac{a_2 b_1}{a_1 c_1}, & c_1 = c_2, \quad b_1 = b_2,\end{cases} \tag{15}$$

ODE (13) possesses the exact solution

$$\phi(v) = \frac{a}{b}\left(1 + c\exp\left(\pm\sqrt{\frac{a}{6}}v\right)\right)^{-2}, \tag{16}$$

where $\alpha = \frac{5}{\sqrt{6}}\sqrt{a}$ and c is an arbitrary constant, [12].

If $c > 0$, then taking into account formulae (12) and (6) and fixing the upper sign in (16), the solution of DLVS (10)

$$\begin{aligned}u_e &= \frac{a}{4b}\left(1 - \tanh\left(\sqrt{\frac{a}{24}}x - \frac{5a}{12}t\right)\right)^2,\\ u_d &= \beta_0 + \beta_1 u_e,\end{aligned} \tag{17}$$

is obtained where the coefficients a, b, β_0 and β_1 are defined by (14) and (15).

If $c < 0$, then solution (16) generates the solution of DLVS (10) as

$$\begin{aligned}u_e &= \frac{a}{4b}\left(1 - \coth\left(\sqrt{\frac{a}{24}}x - \frac{5a}{12}t\right)\right)^2,\\ u_d &= \beta_0 + \beta_1 u_e.\end{aligned} \tag{18}$$

We apply the solutions (17) for solving the Neumann boundary value problem (BVP) corresponding to the nonlinear DLVS (10). Accordingly, a bounded exact solution of the nonlinear BVP, which consists of DLVS (10) (with $\lambda_1 = \lambda_2 = 1$), the initial conditions

$$\begin{aligned}u_e &= \frac{a}{4b}\left(1 - \tanh\left(\sqrt{\frac{a}{24}}x\right)\right)^2 \equiv (u_e)_0(x),\\ u_d &= \beta_0 + \beta_1 (u_e)_0(x),\end{aligned} \tag{19}$$

and the Neumann conditions $(u_e)_x(t,-\infty) = (u_e)_x(t,+\infty) = (u_d)_x(t,-\infty) = (u_d)_x(t,+\infty) = 0$ at infinity in the domain $\Omega = \{(t,x) \in [0,+\infty) \times (-\infty,+\infty)\}$ has the form (17). In formulae (17) and (19), the coefficients a, b, β_0 and β_1 are defined by (14) and (15). For more details, refer to [13].

In order to provide some chemical and biological interpretations, we note that two essentially different cases occur, namely: $\beta_0 \ne 0$ and $\beta_0 = 0$. If $\beta_0 \ne 0$, then the solution (17) possesses the asymptotical behaviour

$$(u_e, u_d) \to \left(\frac{a_1}{b_1}, 0\right), \quad t \to \infty, \tag{20}$$

provided that $A > \max\{B, C\}$, for $A = \frac{a_1}{a_2}$, $B = \frac{b_1}{b_2}$, and $C = \frac{c_1}{c_2}$. In population dynamics, this indicates an uncompromising competition between the populations of the two species, i.e. u_e and u_d. An increase in population u_e leads to a decrease in species u_d. Eventually, the complete disappearance of species u_d takes place. In the case of the opposite condition $A < \min\{B, C\}$, the competition, in fact, has the same character, but in this case, species u_e eventually disappears, while species u_d dominates.

It should be noted that the significance of the asymptotical behaviour (20), which results in the definitive competition between the two species, is one of the advantages of diffusive Lotka-Volterra equations compared to the non-diffusion setting. In the real world, numerous biological interpretations of this type of the competition can be found, too [21-23].

If $\beta_0 = 0$ (in this case, the restriction $a_1 = a_2 = a$ follows from (14)), then solution (17) possesses the property

$$(u_e, u_d) \rightarrow \left(\frac{a(C-1)}{b_2(C-B)}, \frac{a(1-B)}{c_2(C-B)} \right), \quad t \rightarrow \infty, \tag{21}$$

Restriction $\beta_1 = \frac{b_1 - b_2}{c_2 - c_1} > 0$ must also be satisfied (see (15)), which guarantees that solution (17) is nonnegative. In terms of A , B and C , formula (21) implies either the relation $B > A = 1 > C$ or the relation $C > A = 1 > B$. Solution (17) with property (21) describes the case of a 'soft' competition between two populations, which allows an arbitrarily long (in time) coexistence of species u_e and u_d . Figure. 1 illustrates the process of getting to the analytical solution of diffusive Lotka-Volterra system (5).

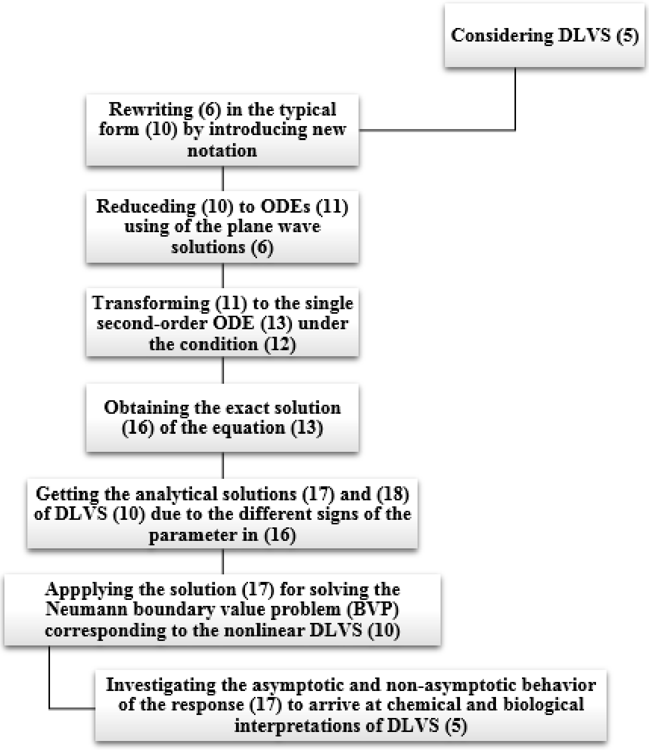

Figure 1. Process of Obtaining the Analytical Solution of Dlvs (5)

More details about the solutions of Von Neumann boundary condition for DLVS (10) with properties (19) and (20) can be found in [24]. Some studies related to the plane wave solutions of Neumann problem for DLVS (10) (in some special cases) can be found in [12-13], [15], [25-26].

VI. CONCLUSION

The results obtained from the analytical solution of the diffusive Lotka-Volterra system using advanced geometrical concepts of Q-conditional symmetry was presented. Using these concepts, the diffusive Lotka-Volterra system was reduced to a set of ordinary differential equations and some examples of their exact solutions along with their biological interpretations were exhibited with the focus on the solutions that expressed different competition scenarios between the two populations.

To give a more detailed chemical and biological interpretation of the responses, both the plate wave solutions and the Neumann boundary value problem answers were obtained for the nonlinear diffusive Lotka-Volterra system. Eventually, the chemical and biological interpretation of deterministic competition and soft competition were shown. The classifications provided for the geometric use of the concepts of extended Lie symmetries and ultimately the analytical solution of the nonlinear diffusive Lotka-Volterra system was presented in the form of new tables and diagrams.

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How to Cite this Article:

Hasan-Zadeh, A. & Mohammadi-Khanaposhtani, M. (2021). Analytical Solutions of Diffusive Lotka-Volterra System Taking into Account the Chemical and Biological Interpretations. International Journal of Science and Engineering Investigations (IJSEI), 10(110), 9-16. http://www.ijsei.com/papers/ijsei-1011021-02.pdf