Assessment of an Analytical Approach in Solving Two Strongly Boundary Value Problems

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Abstract- In this research, a powerful analytical method called Reconstruction of Variational Method (RVIM) is introduced to handle two boundary value problems. One is a parameterized sixth order boundary value problem and the other is a nonlinear boundary value problem arising in the study of thin film flow of a third grade fluid down an inclined plane. With similarity method, the governing equations can be reduced to a system of nonlinear ordinary differential equations. The effectiveness of the method, which is independent of the small parameter, is investigated by comparing the results obtained with the numerical ones (4th order Runge-Kutta method) and the exact ones. For the second problem the velocity profile is plotted and the effect of varying the material constant on the velocity profile is studied.

Keywords- Rotating disk; Condensation film; Analytical approach; Reconstruction of Variational Method, RVIM

I. INTRODUCTION

Nonlinear problems and phenomena play an important role in applied mathematics, physics, engineering and other branches of science. Except for a limited number of these problems, most of them do not have precise analytical solutions; therefore, these nonlinear equations should be solved using approximation methods.

The perturbation method is one well-known method to solve nonlinear equations. But since, using the common perturbation method is based on the existence of a small parameter, developing the method for different application is difficult. Therefore, many different new techniques have been recently introduced to eliminate the small parameter, such as the Adomian decomposition method [1-3], the Homotopy analysis method [4-6], the Homotopy perturbation method [7-10].

In this letter we employ a new and effective analytical method named Reconstruction of Variational Iteration method (RVIM) to solve two strongly boundary value problems. By applying Laplace Transform, RVIM overcomes the difficulty of the perturbation techniques and other variational methods in case of using small parameters and Lagrange multipliers, respectively. Reducing the size of calculations and omitting the difficulty arising in calculation of nonlinear intricately terms are other advantages of this method. Besides, it provides us with a simple way to ensure the convergence of solution series, so that we can always get accurate enough approximations even in first orders of the result iteration.

II. RVIM METHOD AND CONVOLUTION THEOREM

In this section, an alternative method for finding the optimal value of the Lagrange multiplier by the use of the Laplace transform will be introduced [11-12]. Suppose \( x \) is the independent variable; applying Laplace transform to \( u(x,t) \) with respect to \( x \) as variable, we have

\[
\mathcal{L}\{u(x,t);s\} = \int_{0}^{\infty} e^{-st}u(x,t)dt
\]

\[
U(s) = \mathcal{L}\{u(x);s\}
\]

We often come across functions which are not the transform of known functions. But, by means of the convolution theorem, we can take the inverse Laplace transform. The convolution of \( u(x) \) and \( v(x) \) is written as \( u(x) * v(x) \). It is defined as the integral of the functions after one is reversed and shifted. If \( U(s) \) and \( V(s) \) are the Laplace transforms of \( u(x) \) and \( v(x) \), respectively. Then \( U(s) * V(s) \) is the Laplace Transform of \( \int_{0}^{\infty} u(x - \xi) \cdot v(\xi)d\xi \) so by taking inverse Laplace Transform as below,

\[
\mathcal{L}^{-1}\{U(s) * V(s)\} = \int_{0}^{\infty} u(x - \xi) \cdot v(\xi)d\xi
\]

To illustrate the concept of the RVIM, we consider the following general differential equation

\[
L(u(x)) + N(u(x)) = f(x)
\]
Where $L$ and $N$ are linear and nonlinear operators respectively and $f(x)$ is the forcing term. To facilitate our discussion of RVIM, with introducing the new function $h(u(x)) = f(x) - N(u(x))$ and considering the new equation, Eq. (4) can be rewritten as,

$$L(u(x)) = h(u, x)$$

(5)

Now, for implementation the RVIM technique based on new idea of Laplace transform, we apply Laplace Transform on both sides of the Eq. (5). Introducing all artificial initial conditions as zero for the main problem, the left side of the equation after transformation will be

$$\mathcal{L}\{L(u(x))\} = U(s)P(s)$$

(6)

Where $P(s)$ is polynomial with the highest order derivative of the selected linear operator

$$\mathcal{L}\{L(u(x))\} = U(s)P(s) = \mathcal{L}\{h(u, x)\}$$

(7)

$$U(s) = \frac{\mathcal{L}\{h(u, x)\}}{P(s)}$$

(8)

Suppose that $D(s) = \frac{1}{P(s)}$ and $\mathcal{L}\{h(u, x)\} = H(s)$. Using the convolution theorem we have;

$$U(s) = D(s) \cdot H(s) = \mathcal{L}\{d(x) * h(u, x)\}$$

(9)

Taking the inverse Laplace transform on both sides of Eq. (9)

$$u(x) = \int_0^x d(x - \varepsilon) \cdot h(u, \varepsilon) d \varepsilon$$

(10)

Thus the following reconstructed method of variational iteration formula can be obtained

$$u_{n+1}(x) = u_n(x) + \int_0^x d(x - \varepsilon) \cdot h(u_n, \varepsilon) d \varepsilon$$

(11)

III. A PARAMETERIZED SIXTH ORDER BOUNDARY VALUE

A. Governing Equation

Consider the following problem

$$u^{(6)}(x) = (1+c)u^{(4)}(x) - cu^*(x) + cx, \quad 0 < x < 1,$$

(12)

With boundary conditions:

$$u(0) = 1, \quad u'(0) = 1, \quad u''(0) = 0,$$

$$u(L) = \frac{7}{6} + \sinh(L), \quad u'(L) = \frac{1}{2} + \cosh(L), \quad u''(L) = 1 + \sinh(L).$$

(13)

The exact solution of this problem is:

$$u(x) = 1 + \frac{1}{6}x^3 + \sinh(x).$$

(14)

We see the exact solution of this problem does not depend on the parameter $c$ but the problem itself does. This can be viewed by rewriting Eq.(12) as

$$u^{(6)}(x) - u^{(4)}(x) - c[u^{(4)}(x) + u^*(x) + x] = 0,$$

(15)

Which shows that, no matter what the value of $c$ is, a solution of fourth order problem is also a solution of the sixth-order problem.

B. Implementation of RVIM

Considering the initial approximations for $u$ as follow

$$u_0(x) = ax^5 + bx^4 + cx^3 + x + 1,$$

(16)

Rewriting equations (12), based on selective linear operator we have

$$u^{(6)}(x) = (1+c)u^{(4)}(x) - cu^*(x) + cx,$$

(17)

Now Laplace transform is implemented with respect to independent variable $x$ on both sides of Eqs. (17). Using the new artificial boundary conditions (which all of them are zero) we have

$$s^6 U(s) = L\{[(1+c)u^{(4)}(x) - cu^*(x) + cx]\},$$

(18)

By using the Laplace inverse Transform and convolution theorem, it is concluded that

$$u(x) = \frac{1}{120} \int_0^1 (x - \varepsilon)^5 \left((1+c)u^{(4)}(\varepsilon) - cu^*(\varepsilon) + cx\right) d \varepsilon$$

(19)

Hence, we arrive at the following iterative formula for the approximate solution of (12), subject to the boundary conditions (13),

$$u_{n+1}(x) = u_n(x) + \frac{1}{120} \int_0^1 (x - \varepsilon)^5 \left((1+c)u_n^{(4)}(\varepsilon) - cu_n^*(\varepsilon) + cx\right) d \varepsilon$$

(20)

According to above equations, for first order approximation we have:

$$u_1(x) = u_0(x) + \frac{1}{120} \int_0^1 (x - \varepsilon)^5 \left((1+c)u_0^{(4)}(\varepsilon) - cu_0^*(\varepsilon) + cx\right) d \varepsilon$$

(21)

With substituting boundary conditions (13) in the iterative formula (20) the unknown constants of the initial approximation (16) will be determined and with pitting them in the result formula the final answer is approached. More the order of the iteration growth more the accuracy of the solution increases. We calculated the first order of approximation for $c = 1000:$

$$u_1(x) = 1 + 0.00001516539800x^{11}$$

$$-0.0002562231482x^{10} + 0.00407918271x^9$$

$$+0.02306008083x^8 - 0.09516837381x^7$$

$$+0.09623299760x^6 + x + 0.05044617900x^5$$

$$-0.03874093580x^4 + 0.3476013510x^3$$

(22)

In Table 1, results obtained from the first order of RVIM are compared with the exact results (4th order Runge-Kutta method) and the Error value is compared with the Error of two other approximate methods called HAM and OHAM. It is revealed that a good accuracy to the exact results is achieved and RVIM is more accurate and rapid than two other methods, in converging to the exact results.
IV. THIN FILM FLOW PROBLEM

A. Governing Equations

The thin film flow of a third grade fluid down an inclined plane of inclination \( \alpha \neq 0 \) is governed by the following nonlinear boundary value problem [15]

\[
\frac{d^2 u}{dy^2} + \frac{6(\beta_2 + \beta_3)}{\mu} \left( \frac{du}{dy} \right)^2 + \rho g \sin \alpha \frac{\rho}{\mu} = 0 \tag{23}
\]

and

\[
u(0) = 0, \quad \frac{du}{dy} = 0 \quad \text{at} \quad y = \delta. \tag{24}
\]

Introducing the parameters

\[
\beta^* = \frac{3\delta^2 \rho g \sin \alpha}{\mu} (\beta_2 + \beta_3),
\]

The problem in Equations (23) and (24), after omitting asterisks, takes the following form

\[
\frac{d^2 u}{dy^2} + 6\beta \left( \frac{du}{dy} \right)^2 \frac{d^2 u}{dy^2} + 1 = 0, \tag{26}
\]

with

\[
u(0) = 0, \quad \frac{du}{dy} = 0 \quad \text{at} \quad y = 1, \tag{27}
\]

where \( \mu \) is the dynamic viscosity, \( g \) is the gravity, \( \rho \) is the fluid density and \( \beta > 0 \) is the material constant of a third grade fluid. We note that Equation (26) is a second order nonlinear and inhomogeneous differential equation with two boundary conditions; therefore, it is a well-posed problem. Through integration of Equation (26) we have

\[
\frac{du}{dy} + 2\beta \left( \frac{du}{dy} \right)^3 + y = C_1, \tag{28}
\]

where \( C_1 \) is a constant of integration. Employing the second condition of (27) in Equation (28), we obtain \( C_1 = 1 \). Thus, the system (26)-(27) can be written as

\[
\frac{du}{dy} + 2\beta \left( \frac{du}{dy} \right)^3 + (y - 1) = 0, \tag{29}
\]

\[
u(0) = 0 \tag{30}
\]

B. Implementation of RVIM

In this section, we employ RVIM to solve Equation (29). The initial guess is in the following form:

\[
u_0(y) = ay, \tag{31}
\]

Rewriting equations (29), based on selective linear operator we have

\[
\frac{du}{dy} = -2\beta \left( \frac{du}{dy} \right)^3 + (1 - y), \tag{32}
\]

Now Laplace transform is implemented with respect to independent variable von both sides of Eqs. (32). Using the new artificial boundary conditions (which all of them are zero) we have

\[
sU(s) = L \left\{ -2\beta \left( \frac{du}{dy} \right)^3 + (1 - \varepsilon) \right\} \tag{33}
\]

By using the Laplace inverse Transform and convolution theorem, it is concluded that

\[
u(y) = \int_0^y \left\{ -2\beta \left( \frac{du}{dy} \right)^3 + (1 - \varepsilon) \right\} d\varepsilon \tag{34}
\]

Hence, we arrive at the following iterative formula for the approximate solution of (29), subject to the boundary condition (30),

\[
u_{n+1}(y) = \nu_0(y) + \int_0^y \left\{ -2\beta \left( \frac{du}{dy} \right)^3 + (1 - \varepsilon) \right\} d\varepsilon \tag{35}
\]

According to above equations, for first order approximation we have:

\[
u_1(y) = \nu_0(y) + \int_0^y \left\{ -2\beta \left( \frac{du}{dy} \right)^3 + (1 - \varepsilon) \right\} d\varepsilon \tag{36}
\]

With substituting boundary condition (30) in the iterative formula (36) the unknown constant of the initial approximation, \( a \), will be determined and with pitting it in the result formula the final answer is approached. We calculated the second and third order of approximations when \( \beta = 0.1 \):

\[
u_2(y) = y - 0.5x^2 \tag{37}
\]

\[
u_3(y) = 0.9216606811y + 0.02499999999y^4 - 0.09216606814y^3 - 0.3725812384y^2 \tag{38}
\]

\[
u_4(y) = 0.9216989942y - 0.0000099999999984y^3 + 0.000002788066267y^5 - 0.00185904286y^2 + 0.002620995626y^6 + 0.01322497812y^3 - 0.0245140684y^4 - 0.0276857022y^5 - 0.40504871258y^2 \tag{39}
\]

The third and fourth orders are calculated but not mentioned for brevity. All the results presented in figures and
C. Results and Discussion

The effectiveness of Reconstruction of Variational Method (RVIM) is depicted in Table 2 and Figs 1 and 2. The results are well matched with the results carried out by numerical solution (Runge–Kutta). In Table 2, error is introduced as follow:

\[ Error = |u(y)_{NM} - u(y)_{RVIM}| \]  

(40)

Errors of first four orders of the approximation are presented in Table 2. It is obvious that the solution rapidly converges to the exact results and just in the 4th order of the iteration we got efficient accuracy.

Figs. 1 and 2 describes the comparison between the RVIM and numerical results for \( \beta =0.05 \) and \( \beta =0.1 \). An excellent agreement is observed. This accuracy gives high confidence in the validity of this problem, and reveals an excellent agreement in engineering accuracy.
In figure 3 the effects of varying $\beta$ have been investigated. As it is shown by contour in this figure the concentration of $u(y)$ occurs in $x=1$ for small amount of material constant. It is illustrated that with increasing $\beta$ the amount of $u(y)$ decreases generally. In low values of $x$ these changes are not sensible, while in greater $x$ values, the changes of $u(y)$ are more noticeable.

V. CONCLUSION

In this paper, a strong analytical method called Reconstruction of Variational Iteration Method (RVIM) has been successfully applied to find explicit solutions of two boundary value problems, which may occur in different fields of science and engineering. Both of illustrating examples confirm the convenience, reliability and efficiency of this method. RVIM can be introduced to overcome the limitations and difficulties existing in other approximate method, e.g. large computation need, use of small parameters, convergence in high orders of approximations. It is predicted that RVIM can be used widely in mathematical, physical and engineering problems, due to its simplicity and efficiency.

REFERENCES


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